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Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 1

Question:

With respect to an origin O, the position vectors of the points L, M and N are $(4i + 7j + 7k)$, $(i + 3j + 2k)$ and $(2i + 4j + 6k)$ respectively.

(a) Find the vectors ML and MN.

(b) Prove that $\cos \angle LMN = \frac{9}{10}$.

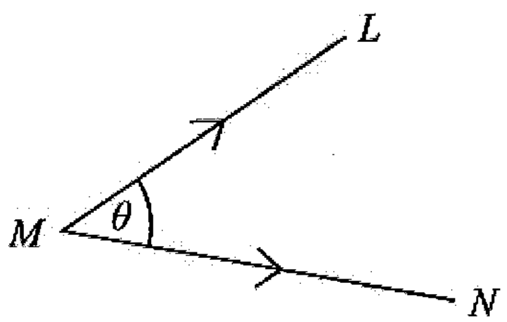
Solution:

$$l = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix}, m = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, n = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

(a) $ML = l - m = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$MN = n - m = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

(b)



$$\cos\theta = \frac{ML \, . \, MN}{|ML| \; |MN|}$$

$$ML \, . \, MN = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 3 + 4 + 20 = 27$$

$$|\,ML\,| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$$
$$|\,MN\,| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18}$$
$$\cos\theta = \frac{27}{\sqrt{50}\sqrt{18}} = \frac{27}{\sqrt{25}\sqrt{2}\sqrt{9}\sqrt{2}} = \frac{27}{5 \times 3 \times 2} = \frac{9}{10}.$$

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Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 2

Question:

The position vectors of the points A and B relative to an origin O are $5i + 4j + k$, $-i + j - 2k$ respectively. Find the position vector of the point P which lies on AB produced such that AP = 2BP.

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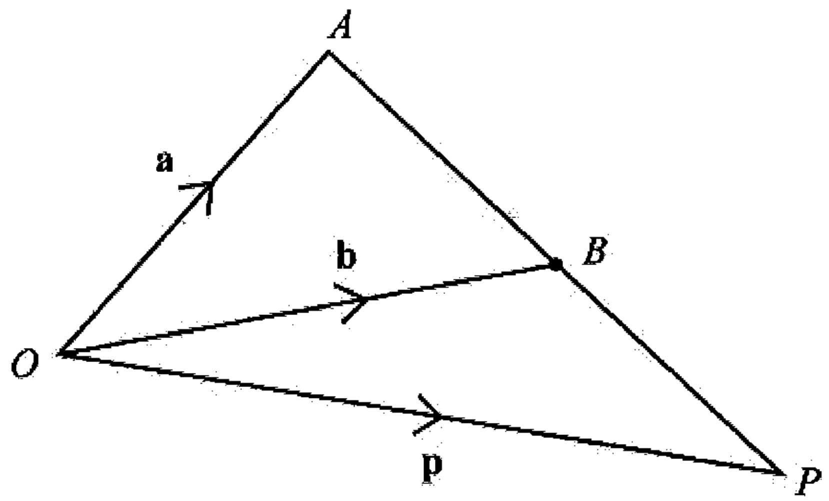
Solution:



$a = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

OP = OA + AP = OA + 2AB

$p = a + 2(b - a) = 2b - a$

$p = 2\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}$

The position vector of P is $-7i - 2j - 5k$.

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 3

Question:

Points A, B, C, D in a plane have position vectors $\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$, $\mathbf{b} = \frac{3}{2}\mathbf{a}$, $\mathbf{c} = 6\mathbf{i} + 3\mathbf{j}$, $\mathbf{d} = \frac{5}{3}\mathbf{c}$ respectively. Write down vector equations of the lines AD and BC and find the position vector of their point of intersection.

E

Solution:

$\mathbf{a} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, $\mathbf{b} = \frac{3}{2}\mathbf{a} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$

$\mathbf{c} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$, $\mathbf{d} = \frac{5}{3}\mathbf{c} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$

Line AD: $\quad AD = \mathbf{d} - \mathbf{a} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Line BC: $\quad BC = \mathbf{c} - \mathbf{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 9 \\ 12 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + s\begin{pmatrix} -3 \\ -9 \end{pmatrix}$

or

$\mathbf{r} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Where AD and BC intersect, $\begin{pmatrix} 6 + 4t \\ 8 - 3t \end{pmatrix} = \begin{pmatrix} 9 + s \\ 12 + 3s \end{pmatrix}$ (Using the last version of BC)

$6 + 4t = 9 + s \quad (\times 3)$

$8 - 3t = 12 + 3s$

$18 + 12t = 27 + 3s$

$8 - 3t = 12 + 3s$

Subtracting: $\quad 10 + 15t = 15$

$\Rightarrow \quad 15t = 5$

$\Rightarrow \quad t = \frac{1}{3}$

Intersection: $\quad \mathbf{r} = \begin{pmatrix} 6 + 4t \\ 8 - 3t \end{pmatrix} = \begin{pmatrix} \frac{22}{3} \\ 7 \end{pmatrix}$

$\mathbf{r} = \frac{22}{3}\mathbf{i} + 7\mathbf{j}$

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Vectors
Exercise K, Question 4

Question:

Find the point of intersection of the line through the points (2, 0, 1) and (− 1 , 3 , 4) and the line through the points (− 1 , 3 , 0) and (4 , − 2 , 5) .
Calculate the acute angle between the two lines.

Solution:

Line through (2, 0, 1) and (− 1 , 3 , 4) .

Let $a = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

$$b - a = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

Equation: $r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$

or $r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Line through (− 1 , 3 , 0) and (4 , − 2 , 5) .

Let $c = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ and $d = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$

$$d - c = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$$

Equation: $r = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$

or $r = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

At the intersection point: $\begin{pmatrix} 2-t \\ t \\ 1+t \end{pmatrix} = \begin{pmatrix} -1+s \\ 3-s \\ s \end{pmatrix}$

$2-t = -1+s$
$t = 3-s$
$1+t = s$
Adding the second and third equations:
$1+2t = 3$
$2t = 2$
$t = 1$
$s = 2$
Intersection point:

$\mathrm{r} = \begin{pmatrix} 2-t \\ t \\ 1+t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ Coordinates (1, 1, 2)

Direction vectors of the lines are $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Calling these **m** and **n**:

$\cos\theta = \dfrac{\mathrm{m}\,.\,\mathrm{n}}{|\mathrm{m}|\ |\mathrm{n}|}$

$\mathrm{m}\,.\,\mathrm{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -1-1+1 = -1$

$|\mathrm{m}| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{3}$
$|\mathrm{n}| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

$\cos\theta = \dfrac{-1}{\sqrt{3}\sqrt{3}} = \dfrac{-1}{3}$

$\theta = 109.5°$ (1 d.p.)
This is the angle between the two vectors.
The acute angle between the lines is $180° - 109.5° = 70.5°$ (1 d.p.).

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 5

Question:

Show that the lines
$r = (-2i + 5j - 11k) + \lambda (3i + j + 3k)$
$r = 8i + 9j + \mu (4i + 2j + 5k)$
intersect. Find the position vector of their common point.

Solution:

$$r = \begin{pmatrix} -2 + 3\lambda \\ 5 + \lambda \\ -11 + 3\lambda \end{pmatrix}, r = \begin{pmatrix} 8 + 4\mu \\ 9 + 2\mu \\ 5\mu \end{pmatrix}$$

At an intersection point: $\begin{pmatrix} -2 + 3\lambda \\ 5 + \lambda \\ -11 + 3\lambda \end{pmatrix} = \begin{pmatrix} 8 + 4\mu \\ 9 + 2\mu \\ 5\mu \end{pmatrix}$

$-2 + 3\lambda = 8 + 4\mu$
$5 + \lambda = 9 + 2\mu \quad (\times 2)$
$-2 + 3\lambda = 8 + 4\mu$
$10 + 2\lambda = 18 + 4\mu$
Subtracting: $-12 + \lambda = -10$
$\Rightarrow \quad \lambda = 12 - 10$
$\Rightarrow \quad \lambda = 2$
$-2 + 6 = 8 + 4\mu$
$\Rightarrow \quad 4\mu = -4$
$\Rightarrow \quad \lambda = -1$
If the lines intersect, $-11 + 3\lambda = 5\mu$:
$-11 + 3\lambda = -11 + 6 = -5$
$5\mu = -5$
The z components are equal, so the lines do intersect. Intersection point:

$$r = \begin{pmatrix} -2 + 3\lambda \\ 5 + \lambda \\ -11 + 3\lambda \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -5 \end{pmatrix} = 4i + 7j - 5k.$$

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Vectors
Exercise K, Question 6

Question:

Find a vector that is perpendicular to both 2i + j − k and i + j − 2k.

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Solution:

Let the required vector be xi + yj + zk.

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$2x + y - z = 0$
$x + y - 2z = 0$
Let $z = 1$:
$2x + y = 1$
$x + y = 2$
Subtracting: $x = -1, y = 3$
So $x = -1, y = 3$ and $z = 1$
A possible vector is $-$i + 3j + k.

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 7

Question:

State a vector equation of the line passing through the points *A* and *B* whose position vectors are i − j + 3k and i + 2j + 2k respectively. Determine the position vector of the point *C* which divides the line segment *AB* internally such that AC = 2CB.

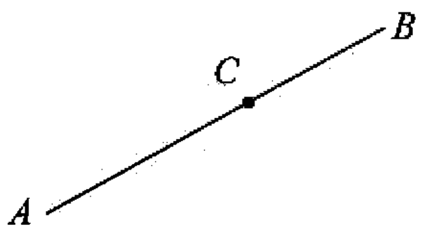
Solution:

$$a = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Equation of line:

$r = a + t (b - a)$

$$r = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$



but AC = 2CB

Position vector of *C*:

$$c = a + \frac{2}{3} \left(b - a \right)$$

$$= \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{7}{3} \end{pmatrix}$$

$$= i + j + \frac{7}{3} k$$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 8

Question:

Vectors **r** and **s** are given by
$r = \lambda i + (2\lambda - 1)j - k$
$s = (1 - \lambda)i + 3\lambda j + (4\lambda - 1)k$
where λ is a scalar.

(a) Find the values of λ for which **r** and **s** are perpendicular.
When $\lambda = 2$, **r** and **s** are the position vectors of the points A and B respectively, referred to an origin O.

(b) Find AB.

(c) Use a scalar product to find the size of $\angle$ BAO, giving your answer to the nearest degree.

E

Solution:

$$r = \begin{pmatrix} \lambda \\ 2\lambda - 1 \\ -1 \end{pmatrix}, \text{ and } \quad s = \begin{pmatrix} 1-\lambda \\ 3\lambda \\ 4\lambda - 1 \end{pmatrix}$$

(a) If **r** and **s** are perpendicular, r . s = 0

$$r \cdot s = \begin{pmatrix} \lambda \\ 2\lambda - 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1-\lambda \\ 3\lambda \\ 4\lambda - 1 \end{pmatrix}$$

$= \lambda(1 - \lambda) + 3\lambda(2\lambda - 1) - 1(4\lambda - 1)$
$= \lambda - \lambda^2 + 6\lambda^2 - 3\lambda - 4\lambda + 1$
$= 5\lambda^2 - 6\lambda + 1$
$\therefore 5\lambda^2 - 6\lambda + 1 = 0$
$(5\lambda - 1)(\lambda - 1) = 0$
$\lambda = \frac{1}{5}$ or $\lambda = 1$

(b) $\lambda = 2$: $\quad a = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 6 \\ 7 \end{pmatrix}$

$$AB = b - a = \begin{pmatrix} -1 \\ 6 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 8 \end{pmatrix}$$

$= -3i + 3j + 8k$

(c) Using vectors AB and AO:

$$AB = \begin{pmatrix} -3 \\ 3 \\ 8 \end{pmatrix}, AO = -a = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

$$\cos \angle BAO = \frac{AB \, . \, AO}{|AB| \; |AO|}$$

$$AB \, . \, AO = \begin{pmatrix} -3 \\ 3 \\ 8 \end{pmatrix} . \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 6 - 9 + 8 = 5$$

$$|AB| = \sqrt{(-3)^2 + 3^2 + 8^2} = \sqrt{82}$$

$$|AO| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = \sqrt{14}$$

$$\cos \angle BAO = \frac{5}{\sqrt{82}\sqrt{14}}$$

$\angle BAO = 82°$ (nearest degree)

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 9

Question:

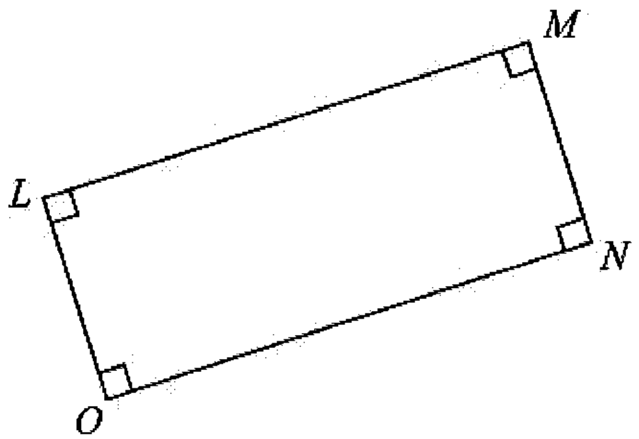
With respect to an origin O, the position vectors of the points L and M are $2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + \mathbf{j} + c\mathbf{k}$ respectively, where c is a constant. The point N is such that $OLMN$ is a rectangle.

(a) Find the value of c.

(b) Write down the position vector of N.

(c) Find, in the form $\mathbf{r} = \mathbf{p} + t\mathbf{q}$, an equation of the line MN.

Solution:



(a) $\mathbf{l} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} 5 \\ 1 \\ c \end{pmatrix}$

$\mathbf{LM} = \mathbf{m} - \mathbf{l} = \begin{pmatrix} 5 \\ 1 \\ c \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ c-3 \end{pmatrix}$

Since OL and LM are perpendicular, OL . LM = O

$\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ c-3 \end{pmatrix} = 0$

$6 - 12 + 3(c - 3) = 0$

$6 - 12 + 3c - 9 = 0$

$3c = 15$

$c = 5$

(b) $\mathbf{n} = \mathbf{ON} = \mathbf{LM} = \begin{pmatrix} 3 \\ 4 \\ c-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$

$\mathbf{n} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(c) The line *MN* is parallel to *OL*.
Using the point *M* and the direction vector **l**:

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 10

Question:

The point A has coordinates $(7, -1, 3)$ and the point B has coordinates $(10, -2, 2)$. The line l has vector equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + \mathbf{k})$, where λ is a real parameter.

(a) Show that the point A lies on the line l.

(b) Find the length of AB.

(c) Find the size of the acute angle between the line l and the line segment AB, giving your answer to the nearest degree.

(d) Hence, or otherwise, calculate the perpendicular distance from B to the line l, giving your answer to two significant figures.

E

Solution:

(a) Line l: $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

Point A is $(7, -1, 3)$

Using $\lambda = 2$, $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}$

So A lies on the line l.

(b) $AB = \sqrt{(10-7)^2 + [-2-(-1)]^2 + (2-3)^2}$
$= \sqrt{3^2 + (-1)^2 + (-1)^2} = \sqrt{11}$

(c) $AB = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 10 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

Angle between the vectors $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$:

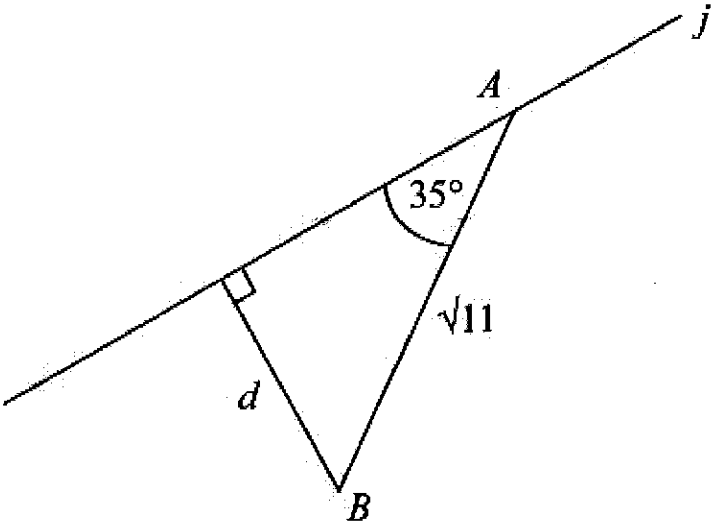
$$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 9 + 1 - 1 = 9$$

The magnitude of each of the vectors is $\sqrt{11}$

So $\cos\theta = \frac{9}{\sqrt{11}\sqrt{11}} = \frac{9}{11}$

$\Rightarrow \quad \theta = 35°$ (nearest degree)

(d)



$\sin 35° = \frac{d}{\sqrt{11}}$

$d = \sqrt{11}\sin 35° = 1.9$ (2 s.f.)

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 11

Question:

Referred to a fixed origin O, the points A and B have position vectors $(5\mathrm{i} - \mathrm{j} - \mathrm{k})$ and $(\mathrm{i} - 5\mathrm{j} + 7\mathrm{k})$ respectively.

(a) Find an equation of the line AB.

(b) Show that the point C with position vector $4\mathrm{i} - 2\mathrm{j} + \mathrm{k}$ lies on AB.

(c) Show that OC is perpendicular to AB.

(d) Find the position vector of the point D, where $D \not\equiv A$, on AB such that $|\mathrm{OD}| = |\mathrm{OA}|$.

E

Solution:

(a) $\mathrm{a} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}, \mathrm{b} = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix}$

$$\mathrm{b} - \mathrm{a} = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$$

Equation of AB:

$$\mathrm{r} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$$

or

$$\mathrm{r} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

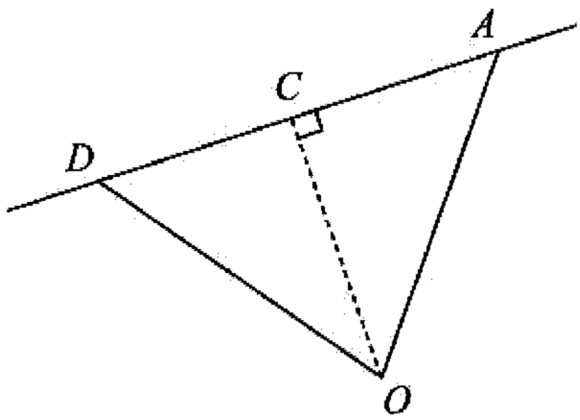
(b) Using $t = 1$: $\mathrm{r} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

So the point with position vector $4\mathrm{i} - 2\mathrm{j} + \mathrm{k}$ lies on AB.

(c) $\text{OC . AB} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix} = -16 + 8 + 8 = 0$

Since the scalar product is zero, *OC* is perpendicular to *AB*.

(d)



Since OD = OA, DC = CA, so DC = CA.

$\text{CA} = \mathbf{a} - \mathbf{c} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\text{DC} = \mathbf{c} - \mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\text{So } \mathbf{d} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$\mathbf{d} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 12

Question:

Referred to a fixed origin O, the points A, B and C have position vectors $(9\mathbf{i}-2\mathbf{j}+\mathbf{k})$, $(6\mathbf{i}+2\mathbf{j}+6\mathbf{k})$ and $(3\mathbf{i}+p\mathbf{j}+q\mathbf{k})$ respectively, where p and q are constants.

(a) Find, in vector form, an equation of the line l which passes through A and B.
Given that C lies on l:

(b) Find the value of p and the value of q.

(c) Calculate, in degrees, the acute angle between OC and AB,
The point D lies on AB and is such that OD is perpendicular to AB.

(d) Find the position vector of D.

Solution:

$$\mathbf{a}=\begin{pmatrix}9\\-2\\1\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}6\\2\\6\end{pmatrix},\ \mathbf{c}=\begin{pmatrix}3\\p\\q\end{pmatrix}$$

(a) $$\mathbf{b}-\mathbf{a}=\begin{pmatrix}6\\2\\6\end{pmatrix}-\begin{pmatrix}9\\-2\\1\end{pmatrix}=\begin{pmatrix}-3\\4\\5\end{pmatrix}$$

Equation of l:

$$\mathbf{r}=\begin{pmatrix}9\\-2\\1\end{pmatrix}+t\begin{pmatrix}-3\\4\\5\end{pmatrix}$$

(b) Since C lies on l,

$$\begin{pmatrix}3\\p\\q\end{pmatrix}=\begin{pmatrix}9\\-2\\1\end{pmatrix}+t\begin{pmatrix}-3\\4\\5\end{pmatrix}$$

$3=9-3t$
$3t=6$
$t=2$

So $p = -2 + 4t = 6$
and $q = 1 + 5t = 11$

(c) $\cos\theta = \dfrac{OC . AB}{|OC|\ |AB|}$

$$OC . AB = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix} . \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = -9 + 24 + 55 = 70$$

$|OC| = \sqrt{3^2 + 6^2 + 11^2} = \sqrt{166}$
$|AB| = \sqrt{(-3)^2 + 4^2 + 5^2} = \sqrt{50}$

$\cos\theta = \dfrac{70}{\sqrt{166}\sqrt{50}}$

$\theta = 39.8°$ (1 d.p.)

(d) If OD and AB are perpendicular, d . (b − a) = 0

Since **d** lies on AB, use $d = \begin{pmatrix} 9-3t \\ -2+4t \\ 1+5t \end{pmatrix}$

$$\begin{pmatrix} 9-3t \\ -2+4t \\ 1+5t \end{pmatrix} . \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = 0$$

$-3(9-3t) + 4(-2+4t) + 5(1+5t) = 0$
$-27 + 9t - 8 + 16t + 5 + 25t = 0$
$50t = 30$
$t = \dfrac{3}{5}$

$$d = \begin{pmatrix} 9-\frac{9}{5} \\ -2+\frac{12}{5} \\ 1+3 \end{pmatrix} = \frac{36}{5}i + \frac{2}{5}j + 4k$$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 13

Question:

Referred to a fixed origin O, the points A and B have position vectors $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and $(5\mathbf{i} - 3\mathbf{j})$ respectively.

(a) Find, in vector form, an equation of the line l_1 which passes through A and B.
The line l_2 has equation $\mathbf{r} = (4\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, where λ is a scalar parameter.

(b) Show that A lies on l_2.

(c) Find, in degrees, the acute angle between the lines l_1 and l_2.
The point C with position vector $(2\mathbf{i} - \mathbf{k})$ lies on l_2.

(d) Find the shortest distance from C to the line l_1.

Solution:

(a) $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

Equation of l_1:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

(b) Equation of l_2:

$$\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

Using $\lambda = -3, \mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

So A lies on the line l_2.

(c) Direction vectors of l_1 and l_2 are $\begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Calling these **m** and **n**:

$\cos\theta = \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}|\ |\mathbf{n}|}$

$\mathbf{m} \cdot \mathbf{n} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 4 + 10 + 6 = 20$

$|\mathbf{m}| = \sqrt{4^2 + (-5)^2 + 3^2} = \sqrt{50}$

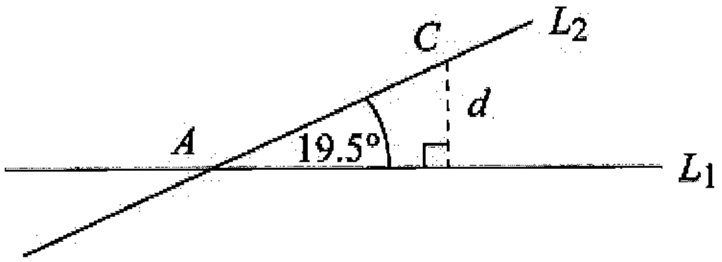
$|\mathbf{n}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

$\cos\theta = \frac{20}{3\sqrt{50}}$

$\theta = 19.5°$ (1 d.p.)

The angle between l_1 and l_2 is $19.5°$ (1 d.p.).

(d) $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$



$|AC| = \sqrt{(2-1)^2 + (0-2)^2 + [-1-(-3)]^2}$
$= \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

$\sin\theta = \frac{d}{AC}$

$d = AC\sin\theta = 3 \times \frac{1}{3} = 1$

The shortest distance from C to l_1 is 1 unit.

5K

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 14

Question:

Two submarines are travelling in straight lines through the ocean. Relative to a fixed origin, the vector equations of the two lines, l_1 and l_2, along which they travel are
r = 3i + 4j − 5k + λ (i − 2j + 2k)
and r = 9i + j − 2k + μ (4i + j − k)
where λ and μ are scalars.

(a) Show that the submarines are moving in perpendicular directions.

(b) Given that l_1 and l_2 intersect at the point A, find the position vector of A.
The point B has position vector 10j − 11k.

(c) Show that only one of the submarines passes through the point B.

(d) Given that 1 unit on each coordinate axis represents 100 m, find, in km, the distance AB.

Solution:

(a) Line l_1: $\quad \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Line l_2: $\quad \mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

Using the direction vectors:

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} . \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 4 - 2 - 2 = 0$$

Since the scalar product is zero, the directions are perpendicular.

(b) At an intersection point: $\begin{pmatrix} 3 + \lambda \\ 4 - 2\lambda \\ -5 + 2\lambda \end{pmatrix} = \begin{pmatrix} 9 + 4\mu \\ 1 + \mu \\ -2 - \mu \end{pmatrix}$

$3 + \lambda = 9 + 4\mu \quad (\times 2)$
$4 - 2\lambda = 1 + \mu$
$6 + 2\lambda = 18 + 8\mu$
$4 - 2\lambda = 1 + \mu$
Adding: $10 = 19 + 9\mu$
$\Rightarrow \quad 9\mu = -9$
$\Rightarrow \quad \mu = -1$
$3 + \lambda = 9 - 4$
$\Rightarrow \quad \lambda = 2$

Intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$

Position vector of A is $\mathbf{a} = 5\mathbf{i} - \mathbf{k}$.

(c) Position vector of B: $\mathbf{b} = 10\mathbf{j} - 11\mathbf{k} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

For l_1, to give zero as the x component, $\lambda = -3$.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$$

So B lies on l_1.

For l_2, to give -11 as the z component, $\mu = 9$.

$$\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + 9\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 45 \\ 10 \\ -11 \end{pmatrix}$$

So B does not lie on l_2.

So only one of the submarines passes through B.

(d) $|AB| = \sqrt{(0-5)^2 + (10-0)^2 + [-11-(-1)]^2}$
$= \sqrt{(-5)^2 + 10^2 + (-10)^2}$
$= \sqrt{225} = 15$

Since 1 unit represents 100 m, the distance AB is
$15 \times 100 = 1500\text{m} = 1.5\text{km}$.