Solutionbank
Edexcel AS and A Level Modular Mathematics

6L

Integration
Exercise L, Question 1

Question:

It is given that $y = x^{\frac{3}{2}} + \frac{48}{x}, x > 0$.

(a) Find the value of x and the value of y when $\frac{dy}{dx} = 0$.

(b) Show that the value of y which you found is a minimum.

The finite region R is bounded by the curve with equation $y = x^{\frac{3}{2}} + \frac{48}{x}$, the lines $x = 1, x = 4$ and the x-axis.

(c) Find, by integration, the area of R giving your answer in the form $p + q \ln r$, where the numbers p, q and r are to be found.

E

Solution:

(a) $y = x^{\frac{3}{2}} + 48x^{-1} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 48x^{-2}$

$\frac{dy}{dx} = 0 \Rightarrow \frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$

$\Rightarrow x^{\frac{5}{2}} = \frac{2}{3} \times 48 = 32$

$\Rightarrow x = 4, y = 2^3 + 12 = 20$

$\Rightarrow x = 4 , y = 20$

(b) $\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + 96x^{-3} > 0$ for all $x > 0$

$\therefore$ 20 is a minimum value of y

(c) Area $= \int_1^4 \left(x^{\frac{3}{2}} + \frac{48}{x} \right) dx$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} + 48\ln|x| \right]_1^4$$

$$= \left(\frac{2}{5} \times 32 + 48\ln 4 \right) - \left(\frac{2}{5} + 0 \right)$$

$$= \frac{62}{5} + 48\ln 4$$

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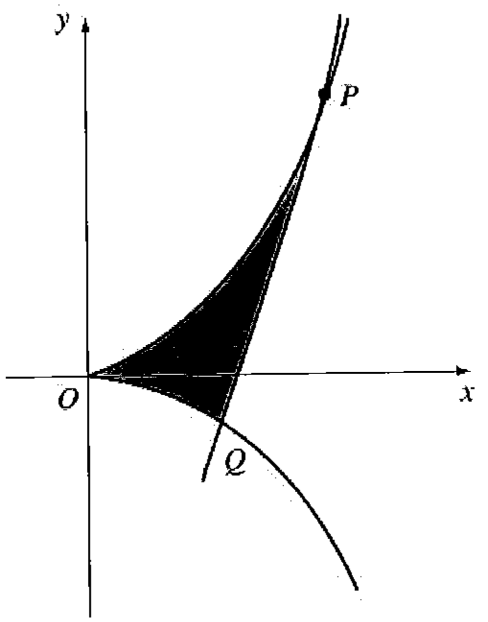
Integration
Exercise L, Question 2

Question:

The curve C has two arcs, as shown, and the equations
$x = 3t^2, y = 2t^3,$
where t is a parameter.



(a) Find an equation of the tangent to C at the point P where $t = 2$.
The tangent meets the curve again at the point Q.

(b) Show that the coordinates of Q are $(3 , - 2)$.
The shaded region R is bounded by the arcs OP and OQ of the curve C, and the line PQ, as shown.

(c) Find the area of R.

E

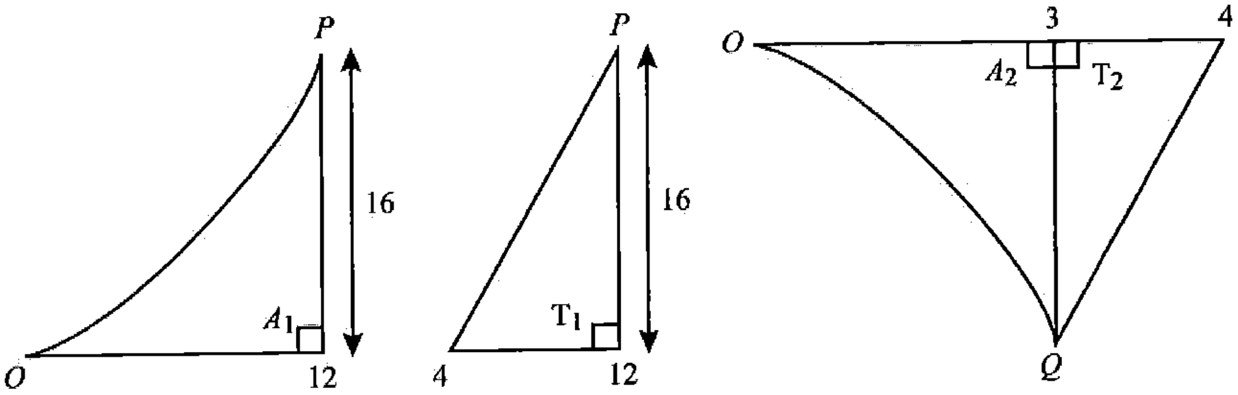
Solution:

(a) $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{6t^2}{6t} = t$

P is (12, 16)
$\therefore$ tangent is $y - 16 = 2 (x - 12)$ or $y = 2x - 8$

(b) Substitute $x = 3t^2, y = 2t^3$ into the equation for the tangent
$\Rightarrow \quad 2t^3 = 6t^2 - 8$
$\Rightarrow \quad t^3 - 3t^2 + 4 = 0$
$\Rightarrow \quad (t - 2) ^2 (t + 1) = 0$
$\Rightarrow \quad t = - 1$ at $Q (3 , - 2)$

(c)



Area of $R = A_1 - T_1 + A_2 + T_2$

$A_1 + A_2 = \int y\,dx = \int_{t=-1}^{t=2} 2t^3 \times 6t\,dt = \int_{-1}^{2} 12t^4\,dt$

$= \left[\frac{12}{5}t^5 \right]_{-1}^{2} = \left(\frac{12 \times 32}{5} \right) - \left(-\frac{12}{5} \right) = 79.2$

$T_1 = \frac{1}{2} \times 16 \times 8 = 64$

$T_2 = \frac{1}{2} \times 1 \times 2 = 1$

$\therefore$ area of $R = 79.2 - 64 + 1 = 16.2$

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Integration
Exercise L, Question 3

Question:

(a) Show that $(1+\sin 2x)^2 \equiv \frac{1}{2}(3+4\sin 2x-\cos 4x)$.

(b) The finite region bounded by the curve with equation $y=1+\sin 2x$, the x-axis, the y-axis and the line with equation $x=\frac{\pi}{2}$ is rotated through 2π about the x-axis.
Using calculus, calculate the volume of the solid generated, giving your answer in terms of π.

E

Solution:

(a) $(1+\sin 2x)^2 = 1+2\sin 2x+\sin^2 2x$

$= 1+2\sin 2x+\frac{1}{2}(1-\cos 4x)$

$= \frac{3}{2}+2\sin 2x-\frac{1}{2}\cos 4x$

$= \frac{1}{2}(3+4\sin 2x-\cos 4x)$

(b) $V=\pi\int y^2\,dx=\pi\int_0^{\frac{\pi}{2}}(1+\sin 2x)^2\,dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}(3+4\sin 2x-\cos 4x)\,dx$

$= \frac{\pi}{2}\left[3x-2\cos 2x-\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left[\left(\frac{3\pi}{2}-2\cos\pi-\frac{1}{4}\sin 2\pi\right)-(0-2-0)\right]$

$= \frac{\pi}{2}\left(\frac{3\pi}{2}+2+2\right)$

$= \frac{\pi}{4}(3\pi+8)$

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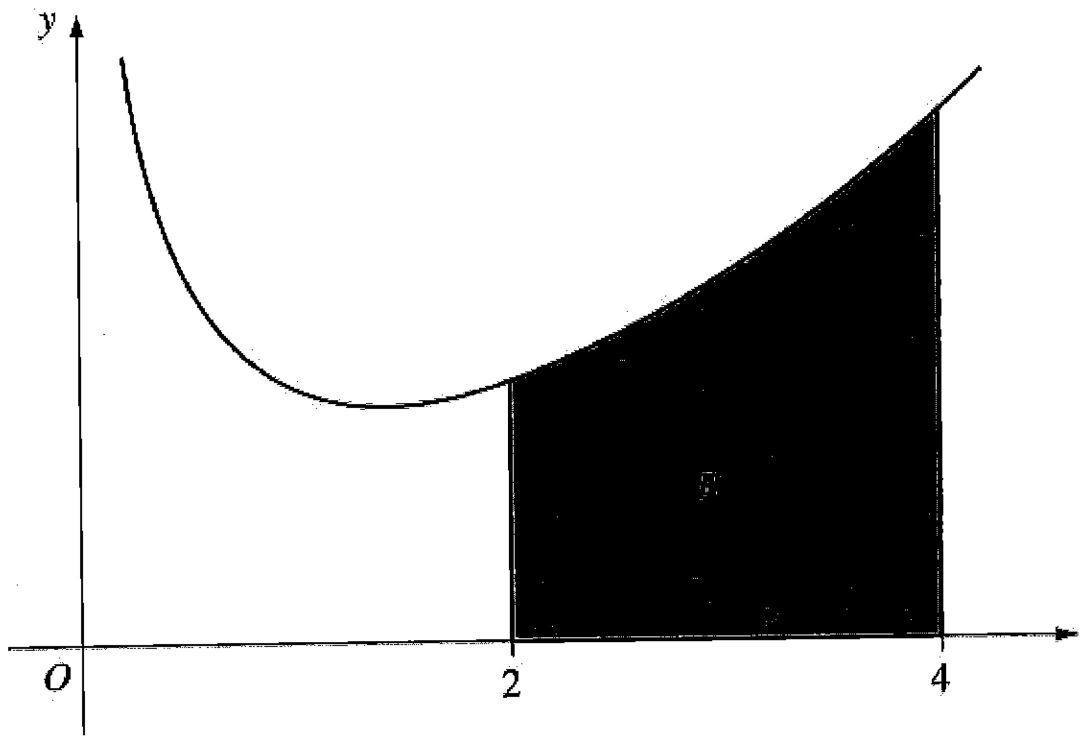
Integration
Exercise L, Question 4

Question:

This graph shows part of the curve with equation $y = f(x)$ where $f(x) \equiv e^{0.5x} + \frac{1}{x}, x > 0$.



The curve has a stationary point at $x = \alpha$.

(a) Find $f'(x)$.

(b) Hence calculate $f'(1.05)$ and $f'(1.10)$ and deduce that $1.05 < \alpha < 1.10$.

(c) Find $\int f(x)\,dx$.
The shaded region R is bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$.

(d) Find, to 2 decimal places, the area of R.

E

Solution:

(a) $f'(x) = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{x^2}$

(b) $f'(1.05) = -0.061... < 0$
$f'(1.10) = +0.040... > 0$
Change of sign $\therefore$ root α in interval (1.05, 1.10)

(c) $\int \left(e^{0.5x} + \frac{1}{x} \right) dx = 2e^{0.5x} + \ln | x | + C$

(d) Area $= \int_2^4 y\,dx$
$= [2e^{0.5x} + \ln | x |]_2^4$
$= (2e^2 + \ln 4) - (2e^1 + \ln 2)$
$= 2e^2 - 2e^1 + \ln 2$
$= 10.03$ (2 d.p.)

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Integration
Exercise L, Question 5

Question:

(a) Find $\int xe^{-x}dx$.

(b) Given that $y = \frac{\pi}{4}$ at $x = 0$, solve the differential equation

$$e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$$

E

Solution:

(a) $I = \int xe^{-x}dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore I = -xe^{-x} - \int(-e^{-x})\,dx$

i.e. $I = -xe^{-x} - e^{-x} + C$

(b) $e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$

$\Rightarrow \quad \int \sin 2y\,dy = \int xe^{-x}dx$

$\Rightarrow \quad -\frac{1}{2}\cos 2y = -xe^{-x} - e^{-x} + C$

$x = 0, y = \frac{\pi}{4} \quad \Rightarrow \quad 0 = 0 - 1 + C \quad \Rightarrow \quad C = 1$

$\therefore \frac{1}{2}\cos 2y = xe^{-x} + e^{-x} - 1$

or $\cos 2y = 2(xe^{-x} + e^{-x} - 1)$

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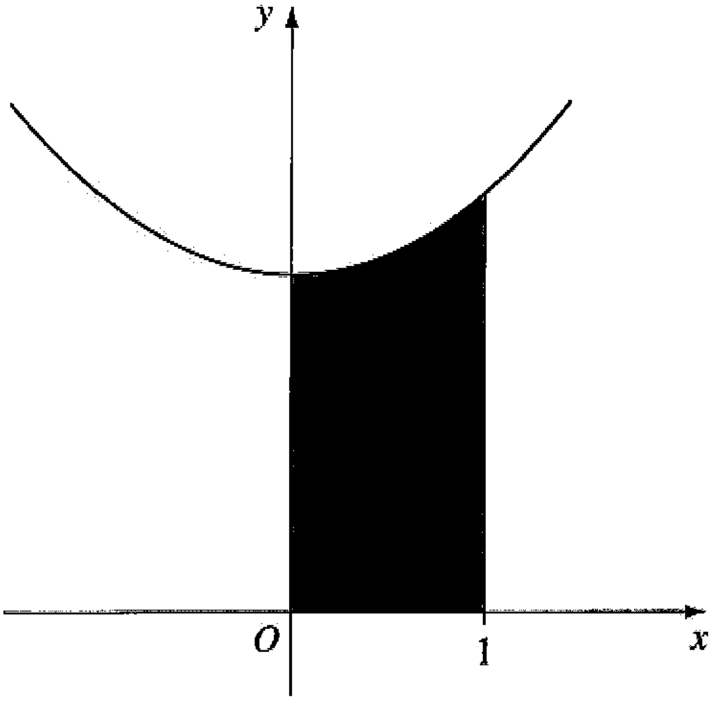
Integration
Exercise L, Question 6

Question:

The diagram shows the finite shaded region bounded by the curve with equation $y = x^2 + 3$, the lines $x = 1$, $x = 0$ and the x-axis. This region is rotated through 360° about the x-axis.
Find the volume generated.



Solution:

$V = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 \left(x^2 + 3 \right)^2 dx$

$= \pi \int_0^1 \left(x^4 + 6x^2 + 9 \right) dx$

$= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_0^1$

$= \pi \left[\left(\frac{1}{5} + 2 + 9 \right) - \left(0 \right) \right]$

$= \frac{56\pi}{5}$

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Integration
Exercise L, Question 7

Question:

(a) Find $\int \frac{1}{x(x+1)} \mathrm{d}x$

(b) Using the substitution $u = \mathrm{e}^x$ and the answer to **a**, or otherwise, find $\int \frac{1}{1+\mathrm{e}^x} \mathrm{d}x$.

(c) Use integration by parts to find $\int x^2 \sin x \, \mathrm{d}x$.

E

Solution:

(a) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

$\therefore \int \frac{1}{x(x+1)} \mathrm{d}x = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) \mathrm{d}x$

$= \ln |x| - \ln |x+1| + C$

$= \ln \left| \frac{x}{x+1} \right| + C$

(b) $I = \int \frac{1}{1+\mathrm{e}^x} \mathrm{d}x \quad u = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x \mathrm{d}x$

$\therefore I = \int \frac{1}{(1+u)} \times \frac{1}{u} \mathrm{d}u = \ln \left| \frac{u}{1+u} \right| + C$ or $\ln \left| \frac{\mathrm{e}^x}{1+\mathrm{e}^x} \right| + C$

(c) $I = \int x^2 \sin x \, \mathrm{d}x$

$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$

$\therefore I = -x^2 \cos x - \int (-\cos x) \times 2x \, \mathrm{d}x$

$= -x^2 \cos x + \int 2x \cos x \, \mathrm{d}x$

Let $J = \int 2x \cos x \, dx$

$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore J = 2x \sin x - \int 2 \sin x \, dx$

$= 2x \sin x + 2 \cos x + C$

$\therefore I = -x^2 \cos x + 2x \sin x + 2 \cos x + k$

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Integration
Exercise L, Question 8

Question:

(a) Find $\int x \sin 2x \, \mathrm{d}x$.

(b) Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x \sin 2x \cos^2 y$.

E

Solution:

(a) $I = \int x \sin 2x \, \mathrm{d}x$

$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \quad \Rightarrow \quad v = \frac{-1}{2} \cos 2x$

$\therefore I = -\frac{1}{2} x \cos 2x - \int \frac{-1}{2} \cos 2x \, \mathrm{d}x$

$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = x \sin 2x \cos^2 y$

$\Rightarrow \quad \int \sec^2 y \, \mathrm{d}y = \int x \sin 2x \, \mathrm{d}x$

$\Rightarrow \quad \tan y = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$

$y = 0, x = \frac{\pi}{4} \quad \Rightarrow \quad 0 = 0 + \frac{1}{4} + C \quad \Rightarrow \quad C = -\frac{1}{4}$

$\therefore \tan y = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x - \frac{1}{4}$

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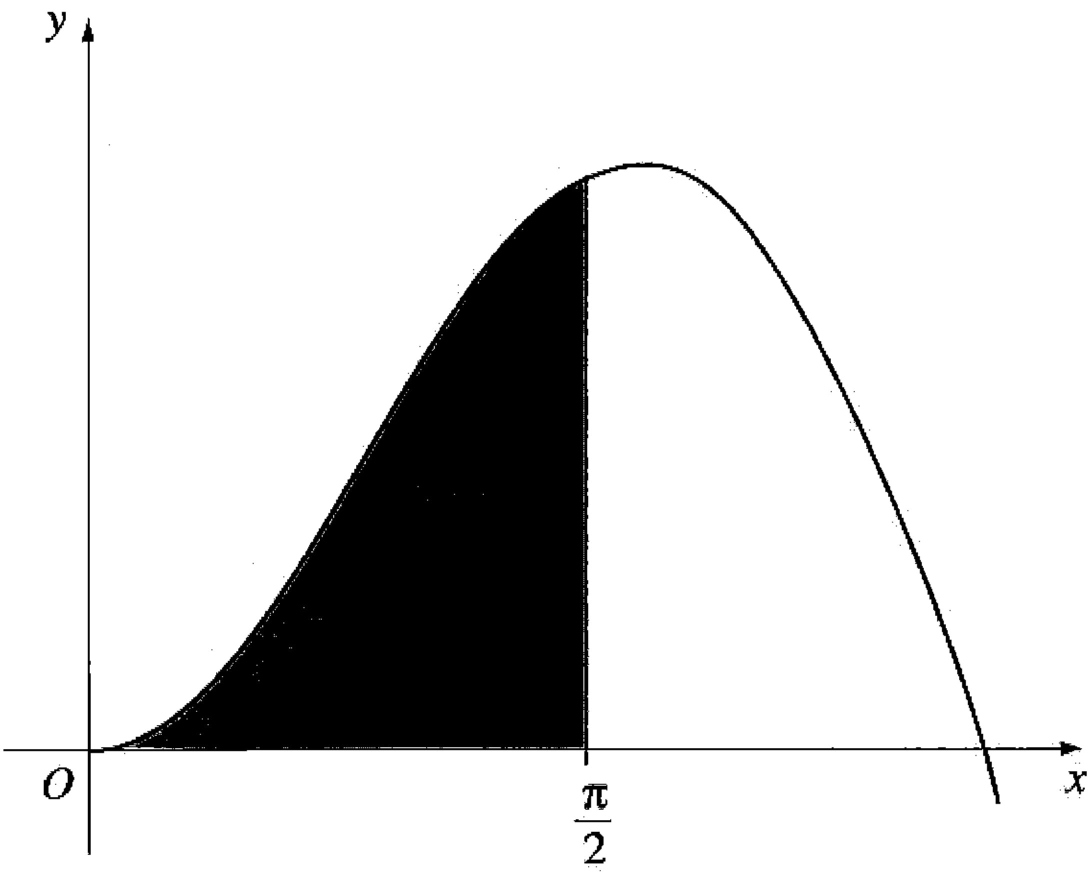
Integration
Exercise L, Question 9

Question:

(a) Find $\int x \cos 2x \, dx$.

(b) This diagram shows part of the curve with equation $y = 2x^{\frac{1}{2}} \sin x$. The shaded region in the diagram is bounded by the curve, the x-axis and the line with equation $x = \frac{\pi}{2}$. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution. Using calculus, calculate the volume of the solid of revolution formed, giving your answer in terms of π.

E



Solution:

(a) $I = \int x \cos 2x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$\frac{dv}{dx} = \cos 2x \quad \Rightarrow \quad v = \frac{1}{2} \sin 2x$

$\therefore I = \frac{x}{2} \sin 2x - \int \frac{1}{2} \sin 2x \, dx$

$= \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C$

(b) $V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx = \pi \int_0^{\frac{\pi}{2}} 4x \sin^2 x \, dx$

$\cos 2A = 1 - 2\sin^2 A \quad \Rightarrow \quad 2\sin^2 x = 1 - \cos 2x$

$\therefore V = \pi \int_0^{\frac{\pi}{2}} 2x \left(1 - \cos 2x \right) dx$

$= \pi \int_0^{\frac{\pi}{2}} 2x \, dx - 2\pi \int_0^{\frac{\pi}{2}} x \cos 2x \, dx$

$= \left[\pi x^2 \right]_0^{\frac{\pi}{2}} - 2\pi \left[\frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^3}{4} - 2\pi \left[\left(\frac{\pi}{4} \sin \pi + \frac{1}{4} \cos \pi \right) - \left(0 + \frac{1}{4} \right) \right]$

$= \frac{\pi^3}{4} + \pi$

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Integration
Exercise L, Question 10

Question:

A curve has equation $y = f(x)$ and passes through the point with coordinates $(0, -1)$. Given that $f'(x) = \frac{1}{2}e^{2x} - 6x$,

(a) use integration to obtain an expression for f(x),

(b) show that there is a root α of the equation $f'(x) = 0$, such that $1.41 < \alpha < 1.43$. E

Solution:

(a) $f'(x) = \frac{1}{2}e^{2x} - 6x$

$\Rightarrow \quad f(x) = \frac{1}{4}e^{2x} - 3x^2 + C$

$f(0) = -1 \Rightarrow -1 = \frac{1}{4} - 0 + C \Rightarrow C = -\frac{5}{4}$

$\therefore f(x) = \frac{1}{4}e^{2x} - 3x^2 - \frac{5}{4}$

(b) $f'(1.41) = -0.07... < 0$
$f'(1.43) = +0.15... > 0$
Change of sign $\therefore$ root in interval (1.41, 1.43).

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Integration
Exercise L, Question 11

Question:

$f(x) = 16x^{\frac{1}{2}} - \frac{2}{x}, x > 0.$

(a) Solve the equation $f(x) = 0$.

(b) Find $\int f(x)\,dx$.

(c) Evaluate $\int_1^4 f(x)\,dx$, giving your answer in the form $p + q\ln r$, where p, q and r are rational numbers.

E

Solution:

(a) $f(x) = 0 \Rightarrow 16x^{\frac{1}{2}} = \frac{2}{x}$

$\Rightarrow 16x^{\frac{3}{2}} = 2$

$\Rightarrow x^{\frac{3}{2}} = \frac{1}{8}$

$\Rightarrow x = \left(\sqrt[3]{\frac{1}{8}}\right)^2 = \frac{1}{4}$

(b) $\int \left(16x^{\frac{1}{2}} - \frac{2}{x}\right) dx = \frac{16x^{\frac{3}{2}}}{\frac{3}{2}} - 2\ln|x| + C$

$= \frac{32}{3}x^{\frac{3}{2}} - 2\ln|x| + C$

(c) $\int_1^4 f(x)\,dx = \left[\frac{32}{3}x^{\frac{3}{2}} - 2\ln|x|\right]_1^4$

$= \left(\frac{32}{3} \times 2^3 - 2\ln 4\right) - \left(\frac{32}{3} - 0\right)$

$$= \frac{224}{3} - 2\ln 4$$

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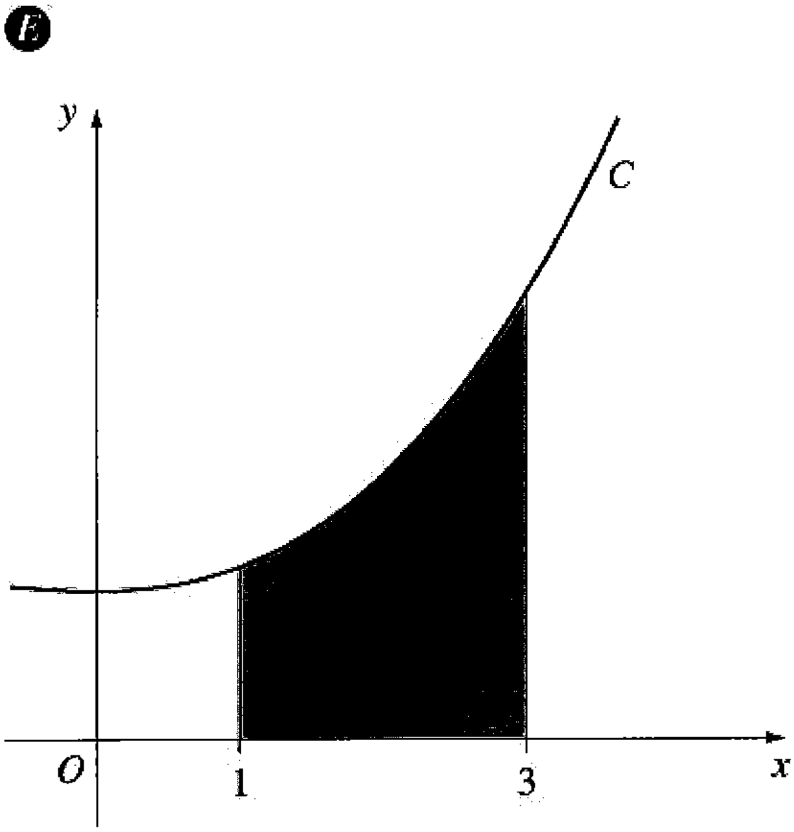
Integration
Exercise L, Question 12

Question:

Shown is part of a curve C with equation $y = x^2 + 3$. The shaded region is bounded by C, the x-axis and the lines with equations $x = 1$ and $x = 3$. The shaded region is rotated through 360° about the x-axis.
Using calculus, calculate the volume of the solid generated. Give your answer as an exact multiple of π.



Solution:

$V = \pi \int_1^3 y^2 \, dx = \pi \int_1^3 (x^2 + 3)^2 \, dx$

$= \pi \int_1^3 (x^4 + 6x^2 + 9) \, dx$

$= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_1^3$

$= \pi \left[\left(\frac{243}{5} + 54 + 27 \right) - \left(\frac{1}{5} + 2 + 9 \right) \right]$

$= \pi \left(\frac{242}{5} + 81 - 11 \right)$

$= 118.4\pi$

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Integration
Exercise L, Question 13

Question:

(a) Find $\int x(x^2+3)^5 \, dx$

(b) Show that $\int_1^e \frac{1}{x^2} \ln x \, dx = 1 - \frac{2}{e}$

(c) Given that $p > 1$, show that $\int_1^p \frac{1}{(x+1)(2x-1)} dx = \frac{1}{3} \ln \frac{4p-2}{p+1}$

E

Solution:

(a) Let $y = (x^2+3)^6$

$\Rightarrow \frac{dy}{dx} = 6(x^2+3)^5 \times 2x$

$\therefore \int x(x^2+3)^5 \, dx = \frac{1}{12}(x^2+3)^6 + C$

(b) $I = \int_1^e \frac{1}{x^2} \ln x \, dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^2} \Rightarrow v = -\frac{1}{x}$

$\therefore I = \left[-\frac{1}{x} \ln x \right]_1^e - \int_1^e \left(-\frac{1}{x^2} \right) dx$

$= \left(-\frac{1}{e} \right) - \left(0 \right) + \left[-\frac{1}{x} \right]_1^e$

$= -\frac{1}{e} + \left(-\frac{1}{e} \right) - \left(-1 \right)$

$= 1 - \frac{2}{e}$

(c) $\frac{1}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1}$

$\Rightarrow \quad 1 \equiv A(2x-1) + B(x+1)$

$x = \frac{1}{2} \quad \Rightarrow \quad 1 = \frac{3}{2}B \quad \Rightarrow \quad B = \frac{2}{3}$

$x = -1 \quad \Rightarrow \quad 1 = -3A \quad \Rightarrow \quad A = -\frac{1}{3}$

$$\therefore \int_1^p \frac{1}{(x+1)(2x-1)}\,dx = \int_1^p \left(\frac{\frac{2}{3}}{2x-1} + \frac{-\frac{1}{3}}{x+1} \right) dx$$

$$= \left[\frac{1}{3}\ln|2x-1| - \frac{1}{3}\ln|x+1| \right]_1^p$$

$$= \left[\frac{1}{3}\ln\left| \frac{2x-1}{x+1} \right| \right]_1^p$$

$$= \left[\frac{1}{3}\ln\left(\frac{2p-1}{p+1} \right) \right] - \left(\frac{1}{3}\ln\frac{1}{2} \right)$$

$$= \frac{1}{3}\ln\left(\frac{4p-2}{p+1} \right)$$

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Integration
Exercise L, Question 14

Question:

$$f\left(x\right) \equiv \frac{5x^2 - 8x + 1}{2x\left(x-1\right)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{\left(x-1\right)^2}$$

(a) Find the values of the constants A, B and C.

(b) Hence find $\int f\left(x\right)\,dx$.

(c) Hence show that $\int_4^9 f\left(x\right)\,dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

Solution:

(a) $f\left(x\right) \equiv \frac{5x^2 - 8x + 1}{2x\left(x-1\right)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{\left(x-1\right)^2}$

$\Rightarrow \quad 5x^2 - 8x + 1 \equiv 2A\left(x-1\right)^2 + 2Bx\left(x-1\right) + 2Cx$

$x = 0 \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

$x = 1 \quad \Rightarrow \quad -2 = 2C \quad \Rightarrow \quad C = -1$

Coefficients of x^2: $\quad 5 = 2A + 2B \quad \Rightarrow \quad B = 2$

(b) $\int f\left(x\right)dx = \int\left(\frac{\frac{1}{2}}{x} + \frac{2}{x-1} - \frac{1}{\left(x-1\right)^2}\right)dx$

$= \frac{1}{2}\ln\left|x\right| + 2\ln\left|x-1\right| + \frac{1}{x-1} + C$

(c) $\int_4^9 f\left(x\right)dx = \left[\frac{1}{2}\ln\left|x\right| + 2\ln\left|x-1\right| + \frac{1}{x-1}\right]_4^9$

$= \left[\ln\left|\sqrt{x}\left(x-1\right)^2\right| + \frac{1}{x-1}\right]_4^9$

$$= \left[\ln \left(3 \times 64 \right) + \frac{1}{8} \right] - \left[\ln \left(2 \times 9 \right) + \frac{1}{3} \right]$$

$$= \ln \left(\frac{3 \times 64}{2 \times 9} \right) + \frac{1}{8} - \frac{1}{3}$$

$$= \ln \frac{32}{3} - \frac{5}{24}$$

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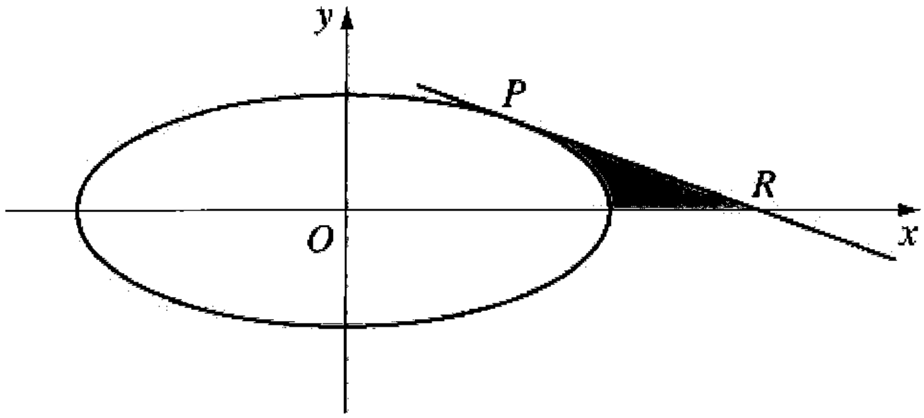
Integration
Exercise L, Question 15

Question:

The curve shown has parametric equations
$x = 5\cos\theta, y = 4\sin\theta, 0 \leq \theta < 2\pi.$



(a) Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

(b) Find an equation of the tangent to the curve at the point P.

(c) Find the coordinates of the point R where this tangent meets the x-axis.
The shaded region is bounded by the tangent PR, the curve and the x-axis.

(d) Find the area of the shaded region, leaving your answer in terms of π.

Solution:

(a) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{4\cos\theta}{5\sin\theta}$

$\therefore$ gradient of tangent at $P = -\frac{4}{5}$

(b) $P = \left(\frac{5}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right)$

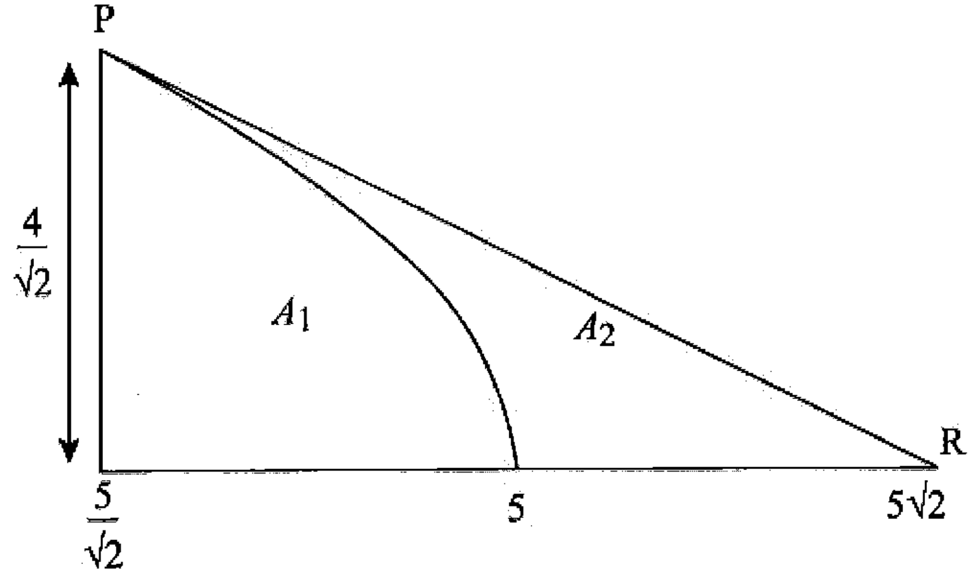
$\therefore$ equation of tangent is

$$y - \frac{4}{\sqrt{2}} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right) \quad \text{or} \quad y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$$

(c) At R, $y = 0 \quad \Rightarrow \quad x = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} = 5\sqrt{2}$

$\therefore R$ is $(5\sqrt{2}, 0)$

(d)



$$A_1 + A_2 = \frac{1}{2} \times \left(5\sqrt{2} - \frac{5}{\sqrt{2}}\right) \times \frac{4}{\sqrt{2}} = \frac{1}{2} \times \frac{5}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 5$$

$$A_1 = \int y\,dx = \int_{\frac{\pi}{4}}^{0} 4\sin\theta \times \left(-5\sin\theta\right) d\theta$$

$$= 10\int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right) d\theta$$

$$= \left[10\theta - 5\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{5\pi}{2} - 5$$

$$\therefore A_2 = 5 - A_1 = 5 - \left(\frac{5\pi}{2} - 5\right) = 10 - 2.5\pi$$

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Integration
Exercise L, Question 16

Question:

(a) Obtain the general solution of the differential equation
$\frac{dy}{dx} = xy^2, y > 0.$

(b) Given also that $y = 1$ at $x = 1$, show that
$y = \frac{2}{3 - x^2}, \; - \sqrt{3} < x < \sqrt{3}$
is a particular solution of the differential equation.

The curve C has equation $y = \frac{2}{3 - x^2}, x \neq - \sqrt{3}, x \neq \sqrt{3}$

(c) Write down the gradient of C at the point (1, 1).

(d) Deduce that the line which is a tangent to C at the point (1, 1) has equation $y = x$.

(e) Find the coordinates of the point where the line $y = x$ again meets the curve C.

E

Solution:

(a) $\frac{dy}{dx} = xy^2$

$\Rightarrow \quad \int \frac{1}{y^2} dy = \int x \, dx$

$\Rightarrow \quad - \frac{1}{y} = \frac{x^2}{2} + C$

or $y = \frac{-2}{x^2 + k} \qquad \left(k = 2C \right)$

(b) $y = 1, x = 1 \quad \Rightarrow \quad 1 = \frac{-2}{1 + k} \quad \Rightarrow \quad k = -3$

$\therefore y = \dfrac{2}{3 - x^2}$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

(c) When $x = 1, y = 1$ $\dfrac{dy}{dx}$ is 1

(d) Equation of tangent is $y - 1 = 1(x - 1)$, i.e. $y = x$.

(e) $x = \dfrac{2}{3 - x^2} \Rightarrow -x^3 + 3x = 2$ or $x^3 - 3x + 2 = 0$

$\Rightarrow (x - 1)^2 (x + 2) = 0$

$\therefore y = x$ meets curve at $(-2, -2)$.

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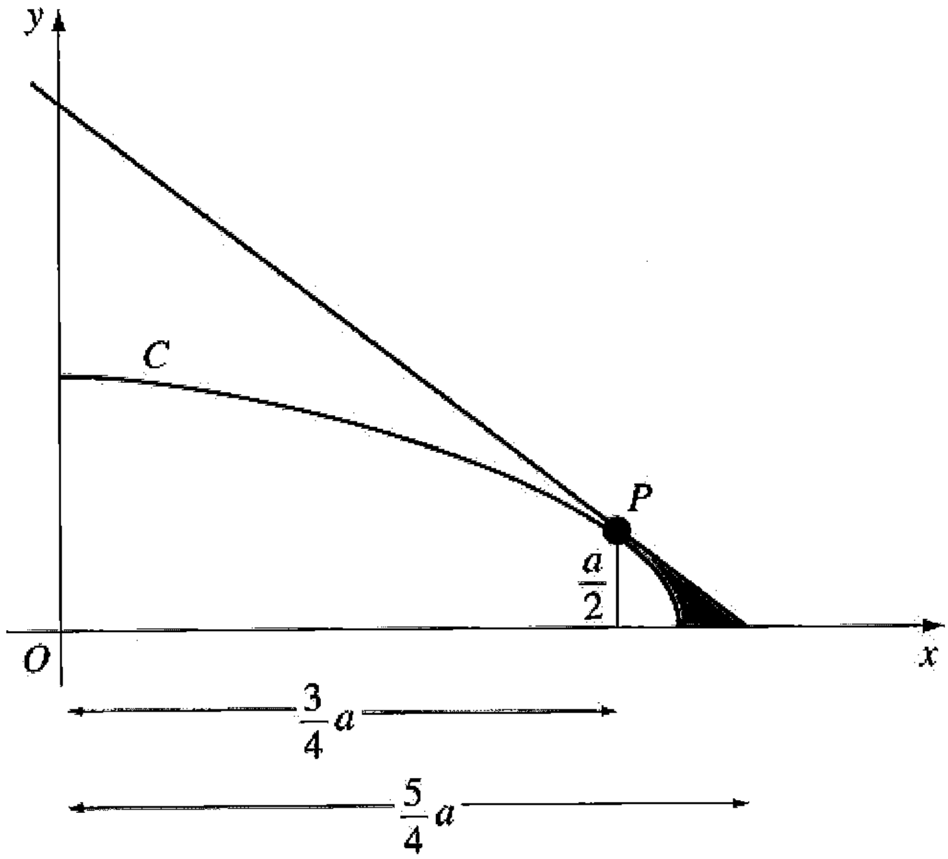
Integration
Exercise L, Question 17

Question:

The diagram shows the curve C with parametric equations

$x = a\sin^2 t, y = a\cos t, 0 \leq t \leq \frac{1}{2}\pi,$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a, \frac{1}{2}a\right)$.



(a) Find $\frac{dy}{dx}$, giving your answer in terms of t.

(b) Find an equation of the tangent to C at P.

(c) Show that a cartesian equation of C is $y^2 = a^2 - ax$.
The shaded region is bounded by C, the tangent at P and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of

revolution.

(d) Use calculus to calculate the volume of the solid revolution formed, giving your answer in the form $k\pi a^3$, where k is an exact fraction. E

Solution:

(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-a\sin t}{2a\sin t\cos t} = -\frac{1}{2}\sec t$

(b) P is $\left(\frac{3}{4}a, \frac{1}{2}a\right)$, so $\cos t = \frac{1}{2}$

$\Rightarrow \quad M = -\frac{1}{2 \times \frac{1}{2}} = -1$

$\therefore$ tangent is $y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a\right)$

or $\quad y = -x + \frac{5}{4}a$

(c) $\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \frac{x}{a} + \frac{y^2}{a^2} = 1$

or $\quad y^2 = a^2 - ax$

(d) volume $=$ cone $- \pi \int_{\frac{3}{4}a}^{a} y^2\,dx$

cone $= \frac{1}{3}\pi\left(\frac{1}{2}a\right)^2\left(\frac{5}{4}a - \frac{3}{4}a\right) = \frac{\pi a^3}{24}$

$\pi \int_{\frac{3}{4}a}^{a} y^2\,dx = \pi\left[a^2x - \frac{a}{2}x^2\right]_{\frac{3}{4}a}^{a}$

$= \pi\left[\left(a^3 - \frac{a^3}{2}\right) - \left(\frac{3}{4}a^3 - \frac{9}{32}a^3\right)\right] = \frac{\pi a^3}{32}$

$\therefore$ Volume $= \pi\left(\frac{a^3}{24} - \frac{a^3}{32}\right) = \frac{\pi a^3}{96}$

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Integration
Exercise L, Question 18

Question:

(a) Using the substitution $u = 1 + 2x$, or otherwise, find

$$\int \frac{4x}{(1+2x)^2} dx, x > -\frac{1}{2},$$

(b) Given that $y = \frac{\pi}{4}$ when $x = 0$, solve the differential equation

$$(1 + 2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$$

E

Solution:

(a) $I = \int \frac{4x}{(1+2x)^2} dx$

$u = 1 + 2x$

$\Rightarrow \quad \frac{du}{2} = dx$ and $4x = 2(u-1)$

$\therefore I = \int \frac{2(u-1)}{u^2} \times \frac{du}{2}$

$= \int \left(\frac{1}{u} - u^{-2} \right) du$

$= \ln|u| + \frac{1}{u} + C$

$= \ln|1 + 2x| + \frac{1}{1+2x} + C$

(b) $(1 + 2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$

$\Rightarrow \quad \int \sin^2 y \, dy = \int \frac{x}{(1+2x)^2} dx$

$\Rightarrow \quad \int 4 \sin^2 y \, dy = \int \frac{4x}{(1+2x)^2} dx$

$\Rightarrow \quad \int (2 - 2\cos 2y) \, dy = I$

$$\Rightarrow \quad 2y - \sin 2y = \ln|1+2x| + \frac{1}{1+2x} + C$$

$$x = 0, y = \frac{\pi}{4} \quad \Rightarrow \quad \frac{\pi}{2} - 1 = \ln 1 + 1 + C$$

$$\Rightarrow \quad C = \frac{\pi}{2} - 2$$

$$\therefore 2y - \sin 2y = \ln|1+2x| + \frac{1}{1+2x} + \frac{\pi}{2} - 2$$

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Integration
Exercise L, Question 19

Question:

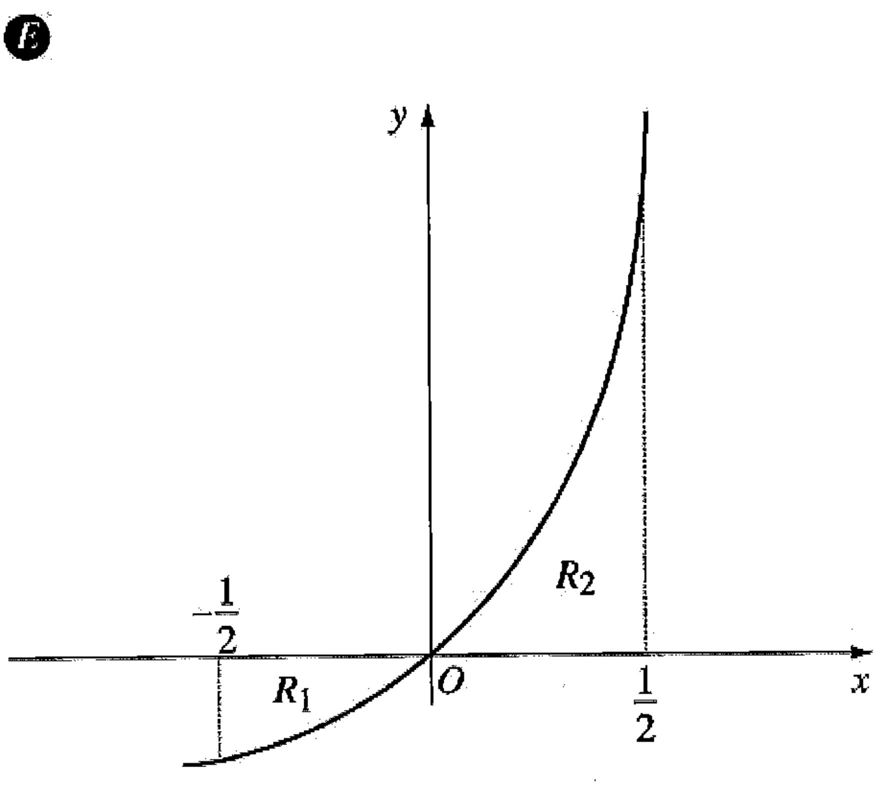
The diagram shows the curve with equation $y = xe^{2x}$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

The finite region R_1 bounded by the curve, the x-axis and the line $x = -\frac{1}{2}$ has area A_1.

The finite region R_2 bounded by the curve, the x-axis and the line $x = \frac{1}{2}$ has area A_2.

(a) Find the exact values of A_1 and A_2 by integration.

(b) Show that $A_1 : A_2 = (e - 2) : e$.



Solution:

(a) $\int xe^{2x}\,dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$$\frac{dv}{dx} = e^{2x} \quad \Rightarrow \quad v = \frac{1}{2}e^{2x}$$

$$\therefore \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$A_1 = -\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_{-\frac{1}{2}}^{0}$$

$$= -\left[\left(0 - \frac{1}{4}\right) - \left(-\frac{1}{4}e^{-1} - \frac{1}{4}e^{-1}\right)\right]$$

$$= \frac{1}{4}\left(1 - 2e^{-1}\right)$$

$$A_2 = \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{4}e^1 - \frac{1}{4}e^1\right) - \left(0 - \frac{1}{4}\right)$$

$$= \frac{1}{4}$$

(b) $$\frac{A_1}{A_2} = \frac{\frac{1}{4}\left(1 - 2e^{-1}\right)}{\frac{1}{4}} = 1 - 2e^{-1} = \frac{e-2}{e}$$

$$\therefore A_1 : A_2 = (e-2) : e$$

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Integration
Exercise L, Question 20

Question:

Find $\int x^2 e^{-x} dx$.

Given that $y = 0$ at $x = 0$, solve the differential equation $\frac{dy}{dx} = x^2 e^{3y - x}$. E

Solution:

$I = \int x^2 e^{-x} dx$

$u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore I = -x^2 e^{-x} - \int (-e^{-x}) \times 2x dx$

$= -x^2 e^{-x} + \int 2x e^{-x} dx$

$J = \int 2x e^{-x} dx$

$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore J = -2x e^{-x} - \int (-e^{-x}) \times 2 dx$

$= -2x e^{-x} - 2e^{-x} + k$

$\therefore I = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

$\frac{dy}{dx} = x^2 e^{3y - x} = x^2 e^{-x} e^{3y}$

$\Rightarrow \quad \int e^{-3y} dy = \int x^2 e^{-x} dx$

$\Rightarrow \quad -\frac{1}{3} e^{-3y} = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

$x = 0, y = 0 \quad \Rightarrow \quad -\frac{1}{3} = -2 + C \quad \Rightarrow \quad C = \frac{5}{3}$

$\therefore \frac{1}{3} e^{-3y} = x^2 e^{-x} + 2x e^{-x} + 2e^{-x} - \frac{5}{3}$

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Integration
Exercise L, Question 21

Question:

The curve with equation $y = e^{3x} + 1$ meets the line $y = 8$ at the point $(h, 8)$.

(a) Find h, giving your answer in terms of natural logarithms.

(b) Show that the area of the finite region enclosed by the curve with equation $y = e^{3x} + 1$, the x-axis, the y-axis and the line $x = h$ is $2 + \frac{1}{3}\ln 7$.

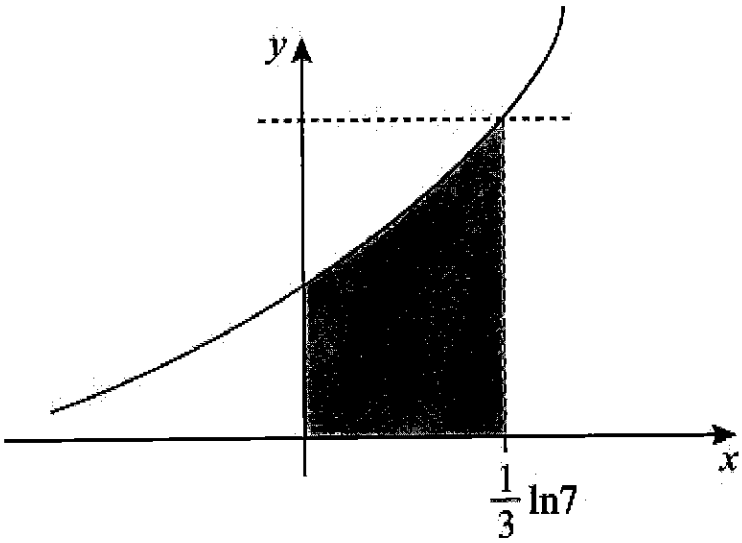
E

Solution:

(a) $8 = e^{3x} + 1 \quad \Rightarrow \quad 7 = e^{3x}$

$\therefore x = \frac{1}{3}\ln 7$, i.e. $h = \frac{1}{3}\ln 7$

(b)



$$\text{Area} = \int_0^{\frac{1}{3}\ln 7} y\,dx$$

$$= \int_0^{\frac{1}{3}\ln 7} \left(e^{3x} + 1 \right) dx$$

$$= \left[\frac{1}{3}e^{3x} + x \right]_0^{\frac{1}{3}\ln 7}$$

$$= \left(\frac{1}{3}e^{\ln 7} + \frac{1}{3}\ln 7 \right) - \left(\frac{1}{3} + 0 \right)$$

$$= \frac{1}{3}\left(7 + \ln 7 \right) - \frac{1}{3}$$

$$= \frac{1}{3}\left(6 + \ln 7 \right)$$

$$= 2 + \frac{1}{3}\ln 7$$

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Integration
Exercise L, Question 22

Question:

(a) Given that

$$\frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x - 1} + \frac{C}{x + 1},$$

find the values of the constants A, B and C.

(b) Given that $x = 2$ at $t = 1$, solve the differential equation

$$\frac{dx}{dt} = 2 - \frac{2}{x^2}, x > 1.$$

You need not simplify your final answer.

Solution:

(a) $\frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x - 1} + \frac{C}{x + 1}$

$\Rightarrow \quad x^2 \equiv A(x - 1)(x + 1) + B(x + 1) + C(x - 1)$

$x = 1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

$x = -1 \quad \Rightarrow \quad 1 = -2C \quad \Rightarrow \quad C = -\frac{1}{2}.$

Coefficients of x^2: $\quad 1 = A \quad \Rightarrow \quad A = 1$

(b) $\frac{dx}{dt} = 2\frac{(x^2 - 1)}{x^2}$

$\Rightarrow \quad \int \frac{x^2}{x^2 - 1} dx = \int 2\, dt$

$\Rightarrow \quad \int \left(1 + \frac{\left(\frac{1}{2}\right)}{x - 1} - \frac{\left(\frac{1}{2}\right)}{x + 1} \right) dx = 2t$

$\Rightarrow \quad x + \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| = 2t + C$

$x = 2, t = 1 \quad \Rightarrow \quad 2 + \frac{1}{2}\ln\frac{1}{3} = 2 + \quad C \quad \Rightarrow \quad C = \frac{1}{2}\ln\frac{1}{3}$

$\therefore x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t + \frac{1}{2}\ln\frac{1}{3}$

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Integration
Exercise L, Question 23

Question:

The curve C is given by the equations

$x = 2t, y = t^2,$

where t is a parameter.

(a) Find an equation of the normal to C at the point P on C where $t = 3$.
The normal meets the y-axis at the point B. The finite region R is bounded by the part of the curve C between the origin O and P, and the lines OB and OP.

(b) Show the region R, together with its boundaries, in a sketch.
The region R is rotated through 2π about the y-axis to form a solid S.

(c) Using integration, and explaining each step in your method, find the volume of S, giving your answer in terms of π.

E

Solution:

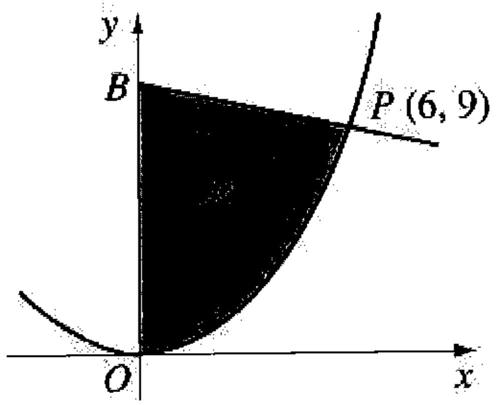
(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t}{2} = t.$

$\therefore$ at P (6, 9) gradient of normal is $-\frac{1}{3}$

$\therefore$ equation of normal is $y - 9 = -\frac{1}{3}\left(x - 6\right)$ or $y = -\frac{1}{3}x + 11$

(b) $x = 2t, y = t^2 \Rightarrow y = \frac{x^2}{4}$

B is (0, 11)



(c) volume $= \text{cone} + \pi \int_0^9 x^2\,dy$

cone $= \frac{1}{3}\pi \times 6^2 \times 2 = 24\pi$

$\pi \int_0^9 x^2\,dy = \pi \int_{t=0}^{t=3} 4t^2 \times 2t\,dt = \pi \int_0^3 8t^3\,dt$

$= \pi \left[2t^4 \right]_0^3 = \pi \times 2 \times 81 = 162\pi$

$\therefore$ Volume of $S = 186\pi$

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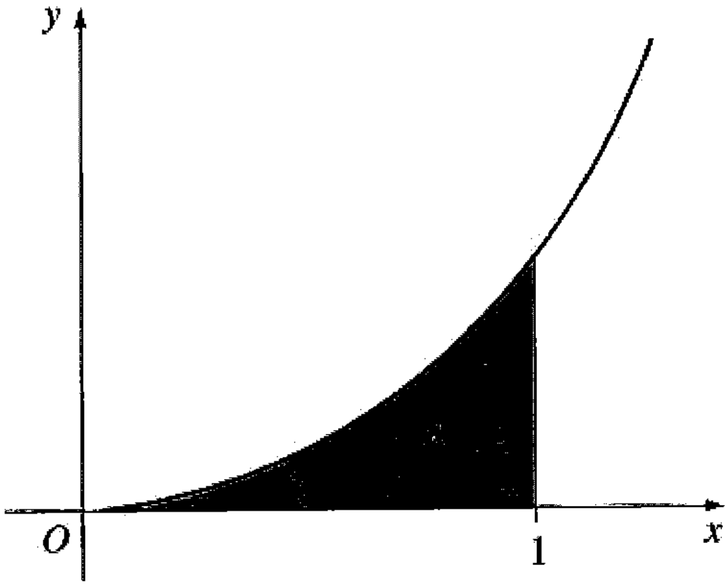
Integration
Exercise L, Question 24

Question:

Shown is part of the curve with equation $y = e^{2x} - e^{-x}$. The shaded region R is bounded by the curve, the x-axis and the line with equation $x = 1$.
Use calculus to find the area of R, giving your answer in terms of e.



Solution:

$$\text{Area} = \int_0^1 \left(e^{2x} - e^{-x} \right) dx$$
$$= \left[\frac{1}{2}e^{2x} + e^{-x} \right]_0^1$$
$$= \left(\frac{1}{2}e^2 + e^{-1} \right) - \left(\frac{1}{2} + 1 \right)$$
$$= \frac{1}{2}\left(e^2 + \frac{2}{e} - 3 \right)$$

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Integration
Exercise L, Question 25

Question:

(a) Given that $2y = x - \sin x \cos x$, show that $\frac{dy}{dx} = \sin^2 x$.

(b) Hence find $\int \sin^2 x \, dx$.

(c) Hence, using integration by parts, find $\int x \sin^2 x \, dx$. E

Solution:

(a) $2y = x - \sin x \cos x$

$$\Rightarrow \quad 2\frac{dy}{dx} = 1 - \left[\cos^2 x + \sin x \left(-\sin x \right) \right] = 1 - \cos^2 x + \sin^2 x$$

$$\therefore \frac{dy}{dx} = \sin^2 x \quad (\text{using } \sin^2 x = 1 - \cos^2 x)$$

(b) $\int \sin^2 x \, dx = y + C_1$

$$= \frac{x}{2} - \frac{1}{2}\sin x \cos x + C_1$$

(c) $\int x \sin^2 x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin^2 x \Rightarrow v = \left(\text{b} \right)$$

$$\therefore \int x \sin^2 x \, dx = \frac{x^2}{2} - \frac{1}{2}x \sin x \cos x - \int \left(\frac{x}{2} - \frac{1}{2}\sin x \cos x \right) dx$$

$$= \frac{x^2}{2} - \frac{1}{2}x \sin x \cos x - \frac{x^2}{4} + \frac{1}{4}\int \sin 2x \, dx$$

$$= \frac{x^2}{4} - \frac{1}{2}x \sin x \cos x - \frac{1}{8}\cos 2x + C_2$$

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Integration
Exercise L, Question 26

Question:

The rate, in $cm^3 s^{-1}$, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, $V cm^3$, in the sump at that instant. At time $t = 0$, $V = A$.

(a) By forming and integrating a differential equation, show that

$V = Ae^{-kt}$

where k is a positive constant.

(b) Sketch a graph to show the relation between V and t.

Given further that $V = \frac{1}{2}A$ at $t = T$,

(c) show that $kT = \ln 2$.

Solution:

(a) $\frac{dv}{dt} = -k\ V$

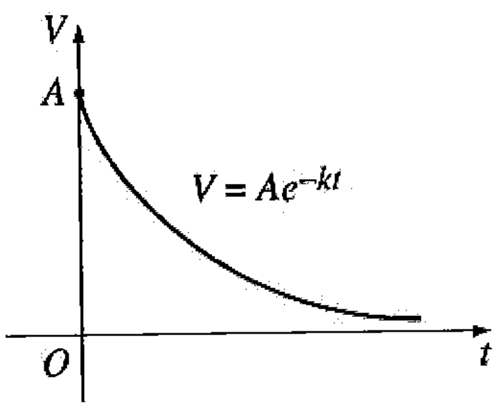
$\Rightarrow \quad \int \frac{1}{V} dV = \int -k dt$

$\Rightarrow \quad \ln|V| = -kt + C$

$\Rightarrow \quad V = A_1 e^{-kt}$

$t = 0, V = A \quad \Rightarrow \quad V = Ae^{-kt} \quad (A_1 = A)$

(b)



(c) $t = T$, $V = \frac{1}{2}A \quad \Rightarrow \quad \frac{1}{2}A = A\mathrm{e}^{-kT}$

$\Rightarrow \quad -\ln 2 = -kT$

$\Rightarrow \quad kT = \ln 2$

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Integration
Exercise L, Question 27

Question:

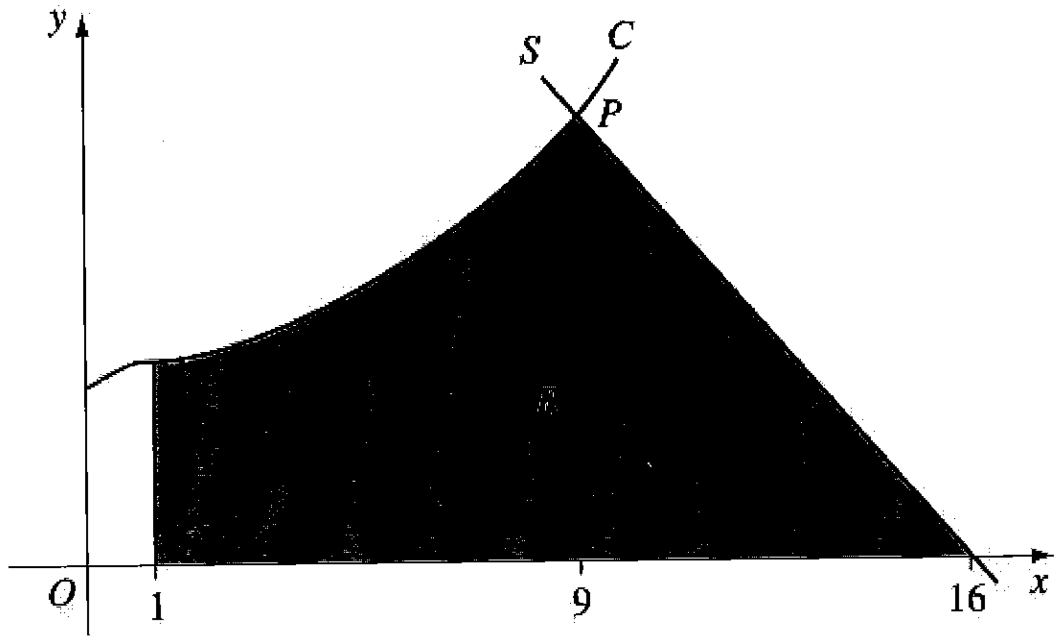
This graph shows part of the curve C with parametric equations

$x = (t+1)^2, y = \frac{1}{2}t^3 + 3, t \geq -1.$

P is the point on the curve where $t = 2$. The line S is the normal to C at P.

(a) Find an equation of S.
The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

(b) Using integration and showing all your working, find the area of R. E



Solution:

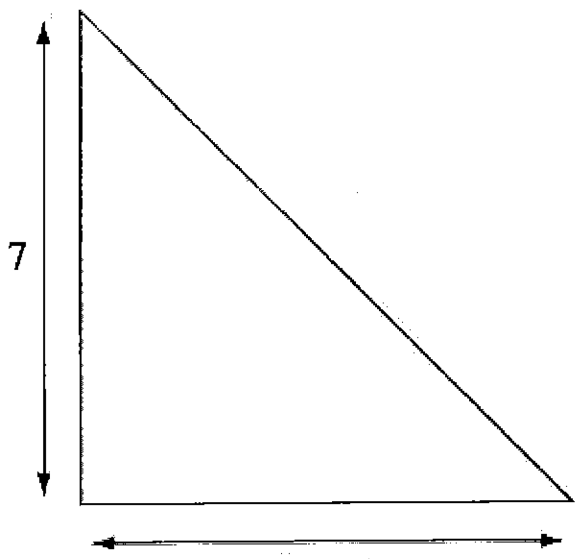
(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{3}{2}t^2}{2(t+1)} = \frac{3t^2}{4(t+1)}$

At P (9, 7) gradient of normal is $-\frac{4 \times 3}{3 \times 2^2} = -1$

$\therefore$ equation of line S is $y - 7 = -1(x - 9)$

i.e. $y = -x + 16$ or $y + x = 16$

(b) Area $= \int_{x=1}^{x=9} y\,dx +$ area of triangle shown below



$$\int_{x=1}^{x=9} y\,dx = \int_{t=0}^{t=2} \left(\frac{1}{2}t^3 + 3\right) \; 2\left(t + 1\right) dt$$

$$= \int_0^2 (t^4 + t^3 + 6t + 6)\,dt$$

$$= \left[\frac{1}{5}t^5 + \frac{1}{4}t^4 + \frac{6t^2}{2} + 6t\right]_0^2$$

$$= \left(\frac{32}{5} + \frac{16}{4} + 3 \times 4 + 6 \times 2\right) - \left(0\right)$$

$$= 34.4$$

$$\therefore \text{ Area } = 34.4 + \frac{1}{2} \times 7^2 = 58.9$$

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Integration
Exercise L, Question 28

Question:

Shown is part of the curve C with parametric equations

$x = t^2, y = \sin 2t, t \geq 0.$

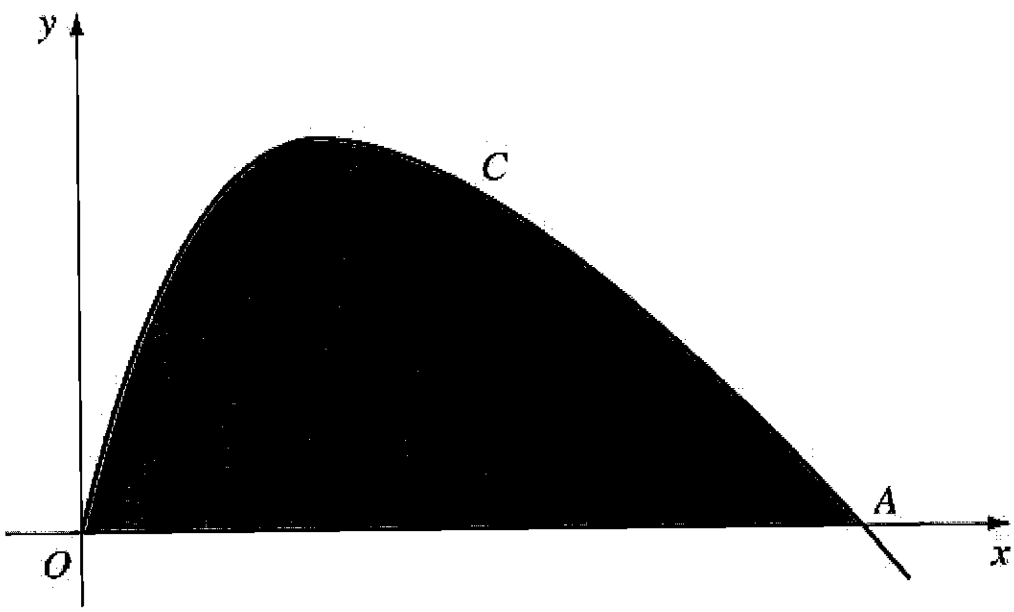
The point A is an intersection of C with the x-axis.

(a) Find, in terms of π, the x-coordinate of A.

(b) Find $\frac{dy}{dx}$ in terms of t, $t > 0$.

(c) Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.
The shaded region is bounded by C and the x-axis.

(d) Use calculus to find, in terms of π, the area of the shaded region. E



Solution:

(a) At A, $y = 0 \Rightarrow \sin 2t = 0 \Rightarrow 2t = 0 \text{ or } \pi \Rightarrow t = \frac{\pi}{2}$

$\therefore A$ is $\left(\left(\frac{\pi}{2} \right)^2, 0 \right)$ or $\left(\frac{\pi^2}{4}, 0 \right)$

(b) $\frac{dy}{dx} = \frac{2\cos 2t}{2t} = \frac{\cos 2t}{t}$

(c) Gradient of tangent at A is $\frac{\cos \pi}{\left(\frac{\pi}{2}\right)} = -\frac{1}{\left(\frac{\pi}{2}\right)} = -\frac{2}{\pi}$

$\therefore$ equation of tangent is $y - 0 = -\frac{2}{\pi}\left(x - \frac{\pi^2}{4}\right)$

$\Rightarrow \quad \pi y = -2x + \frac{2\pi^2}{4}$

or $2\pi y + 4x = \pi^2$

(d) Area $= \int y\,dx = \int_{t=0}^{t=\frac{\pi}{2}} \sin 2t \times 2t\,dt$

$u = t \quad \Rightarrow \quad \frac{du}{dt} = 1$

$\frac{dv}{dt} = 2\sin 2t \Rightarrow v = -\cos 2t$

$\therefore$ Area $= \left[-t\cos 2t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos 2t\right) dt$

$= \left(+\frac{\pi}{2}\right) - \left(0\right) + \left[\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

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Integration
Exercise L, Question 29

Question:

Showing your method clearly in each case, find

(a) $\int \sin^2 x \cos x \, dx$,

(b) $\int x \ln x \, dx$.

Using the substitution $t^2 = x + 1$, where $x > -1$, $t > 0$,

(c) Find $\int \frac{x}{\sqrt{x+1}} dx$.

(d) Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} dx$. (E)

Solution:

(a) Let $y = \sin^3 x \quad \Rightarrow \quad \frac{dy}{dx} = 3 \sin^2 x \cos x$

$\therefore \int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + C$

(b) $\int x \ln x \, dx$

$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \quad \Rightarrow \quad v = \frac{1}{2} x^2$

$\therefore \int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \times \frac{1}{x} dx$

$= \frac{1}{2} x^2 \ln x - \frac{x^2}{4} + C$

(c) $t^2 = x + 1 \quad \Rightarrow \quad 2t \, dt = dx$

$\therefore I = \int \frac{x}{\sqrt{x+1}} dx$

$= \int \frac{t^2 - 1}{t} \times 2t \, dt$

$= \int (2t^2 - 2)\, dt$

$= \frac{2}{3}t^3 - 2t + C$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + C$

$= \frac{2}{3}\sqrt{x+1}\left(x-2\right) + C$

(d) $\int_0^3 \frac{x}{\sqrt{x+1}} dx = \left[\frac{2}{3}\left(x-2\right)\sqrt{x+1}\right]_0^3$

$= \left(\frac{2}{3} \times 2\right) - \left(-\frac{4}{3}\right) = \frac{8}{3}$

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Integration
Exercise L, Question 30

Question:

(a) Using the substitution $u = 1 + 2x^2$, find $\int x (1 + 2x^2) ^5 \mathrm{d}x$.

(b) Given that $y = \frac{\pi}{8}$ at $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x (1 + 2x^2) ^5 \cos^2 2y.$$

Solution:

(a) $u = 1 + 2x^2 \quad \Rightarrow \quad \mathrm{d}u = 4x\,\mathrm{d}x \quad \Rightarrow \quad x\,\mathrm{d}x = \frac{\mathrm{d}u}{4}$

So $\int x (1 + 2x^2) ^5 \mathrm{d}x = \int \frac{u^5}{4} \mathrm{d}u = \frac{u^6}{24} + C_1 = \frac{(1 + 2x^2) ^6}{24} + C_1$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = x (1 + 2x^2) ^5 \cos^2 2y$

$\Rightarrow \quad \int \sec^2 2y\,\mathrm{d}y = \int x (1 + 2x^2) ^5 \mathrm{d}x$

$\Rightarrow \quad \frac{1}{2} \tan 2y = \frac{(1 + 2x^2) ^6}{24} + C_2$

$y = \frac{\pi}{8}, x = 0 \quad \Rightarrow \quad \frac{1}{2} = \frac{1}{24} + C_2 \quad \Rightarrow \quad C_2 = \frac{11}{24}$

$\therefore \tan 2y = \frac{(1 + 2x^2) ^6}{12} + \frac{11}{12}$

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Integration
Exercise L, Question 31

Question:

Find $\int x^2 \ln 2x \, dx$.

E

Solution:

$I = \int x^2 \ln 2x \, dx$

$u = \ln 2x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \quad \Rightarrow \quad v = \frac{x^3}{3}$

$\therefore I = \frac{x^3}{3} \ln 2x - \int \frac{x^3}{3} \times \frac{1}{x} \, dx$

$= \frac{x^3}{3} \ln 2x - \int \frac{x^2}{3} \, dx$

$= \frac{x^3}{3} \ln 2x - \frac{x^3}{9} + C$

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Integration
Exercise L, Question 32

Question:

Obtain the solution of

$x\left(x+2\right)\frac{dy}{dx}=y, y>0, x>0,$

for which $y=2$ at $x=2$, giving your answer in the form $y^2=f(x)$.

E

Solution:

$x\left(x+2\right)\frac{dy}{dx}=y$

$\Rightarrow \quad \int \frac{1}{y}dy=\int \frac{1}{x(x+2)}dx$

$\frac{1}{x(x+2)}\equiv \frac{A}{x}+\frac{B}{x+2}$

$\Rightarrow \quad 1\equiv A(x+2)+Bx$

$x=0 \quad \Rightarrow \quad 1=2A \quad \Rightarrow \quad A=\frac{1}{2}$

$x=-2 \quad \Rightarrow \quad 1=-2B \quad \Rightarrow \quad B=-\frac{1}{2}$

So $\ln y=\int \left(\frac{\left(\frac{1}{2}\right)}{x}-\frac{\left(\frac{1}{2}\right)}{x+2}\right)dx$

$=\frac{1}{2}\ln|x|-\frac{1}{2}\ln|x+2|+C$

$\therefore y=\sqrt{\frac{kx}{x+2}} \quad \left(C=\frac{1}{2}\ln k\right)$

$x=2, y=2 \quad \Rightarrow \quad 2=\sqrt{\frac{2k}{4}} \quad \Rightarrow \quad 4\times 2=k$

$\therefore y=\sqrt{\frac{8x}{x+2}}$ or $y^2=\frac{8x}{x+2}$

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Integration
Exercise L, Question 33

Question:

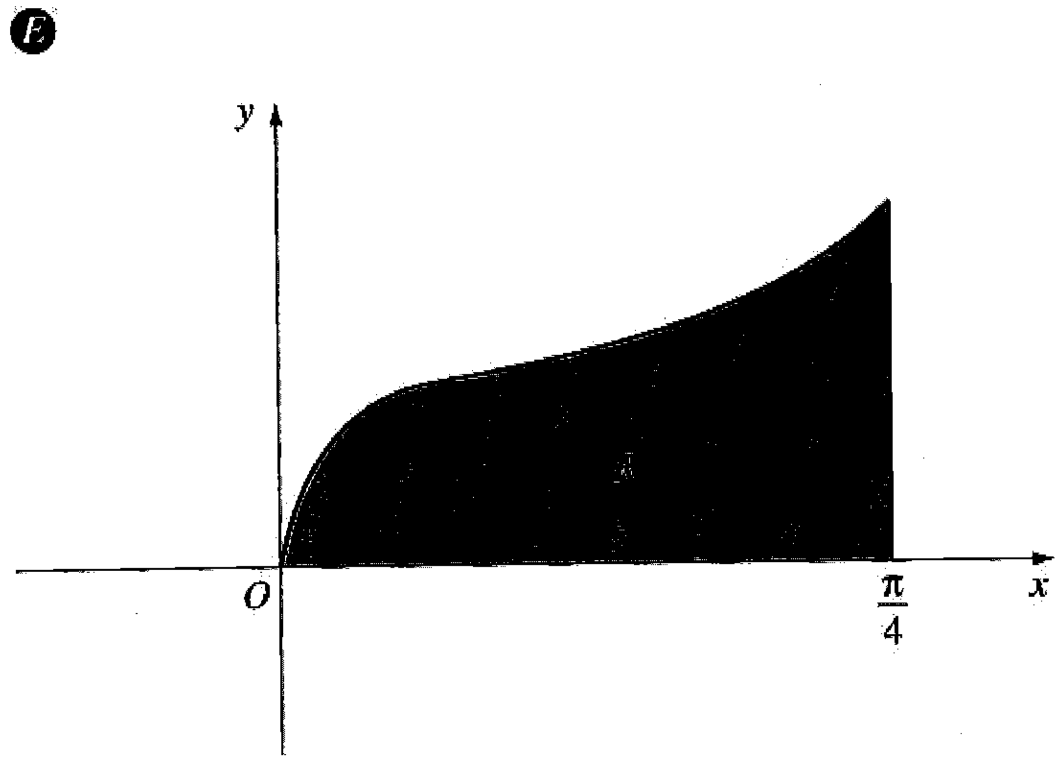
(a) Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = \frac{1}{4}\pi - \frac{1}{2}\ln 2.$$

The finite region R, bounded by the curve with equation $y = x^{\frac{1}{2}} \sec x$, the line $x = \frac{\pi}{4}$ and the x-axis is shown. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid of revolution generated.

(c) Find the gradient of the curve with equation $y = x^{\frac{1}{2}} \sec x$ at the point where $x = \frac{\pi}{4}$.



Solution:

(a) $I = \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \quad \Rightarrow \quad v = \tan x$

$\therefore I = [\,x \tan x\,]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \left(\frac{\pi}{4}\right) - \left(0\right) - [\,\ln|\sec x|\,]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \left[\left(\ln\sqrt{2}\right) - \left(\ln 1\right)\right]$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

(b) $V = \pi \int_0^{\frac{\pi}{4}} y^2 \, dx = \pi \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$

Using (a) $\quad V = \frac{\pi^2}{4} - \frac{\pi}{2}\ln 2 = 1.38$ (3 s.f.)

(c) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \sec x + x^{\frac{1}{2}} \sec x \tan x$

At $x = \frac{\pi}{4}$ $\frac{dy}{dx} = \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \times \sqrt{2} + \frac{\sqrt{\pi}}{2} \times \sqrt{2} \times 1 = \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi}{2}} = 2.05$ (3 s.f.)

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Integration
Exercise L, Question 34

Question:

Part of the design of a stained glass window is shown. The two loops enclose an area of blue glass. The remaining area within the rectangle *ABCD* is red glass. The loops are described by the curve with parametric equations

$x = 3\cos t, y = 9\sin 2t, 0 \leq t < 2\pi.$

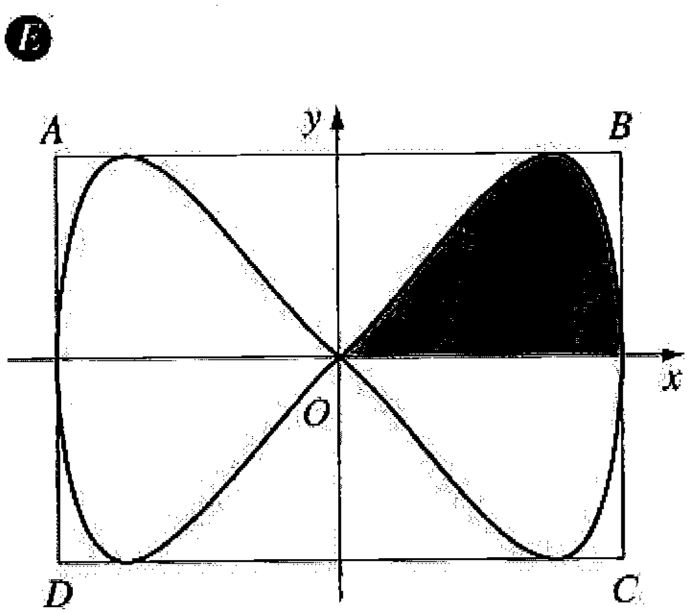
(a) Find the cartesian equation of the curve in the form $y^2 = f(x)$.

(b) Show that the shaded area enclosed by the curve and the x-axis, is given by

$\int_0^{\frac{\pi}{2}} A\sin 2t \sin t \, dt$, stating the value of the constant A.

(c) Find the value of this integral.
The sides of the rectangle *ABCD* are the tangents to the curve that are parallel to the coordinate axes. Given that 1 unit on each axis represents 1 cm,

(d) find the total area of the red glass.



Solution:

(a) $x = 3\cos t$

$y = 9\sin 2t \quad \Rightarrow \quad y = 18\cos t \sin t$

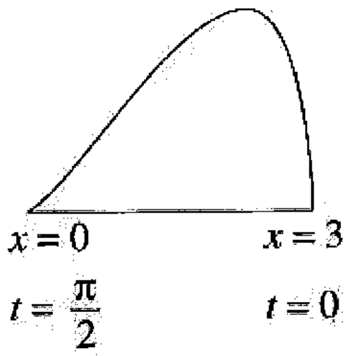
$\Rightarrow \quad y = 6x\sin t$

$\therefore \cos t = \frac{x}{3}, \sin t = \frac{y}{6x}$

$\cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{36x^2} = 1$

i.e. $4x^4 + y^2 = 36x^2$
or $y^2 = 4x^2 (9 - x^2)$

(b)



Area $= \int y \, dx$

$= \int_{t = \frac{\pi}{2}}^{t = 0} 9 \sin 2t \times \left(- 3 \sin t \right) dt$

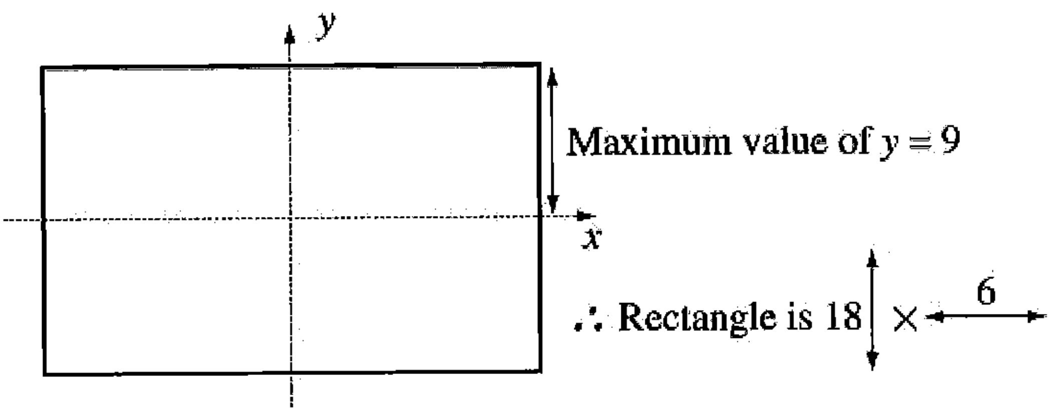
$= 27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt$

(c) $27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt = 54 \int_0^{\frac{\pi}{2}} \sin^2 t \cos t \, dt$

$= \left[\frac{54 \sin^3 t}{3} \right]_0^{\frac{\pi}{2}}$

$= (18 \times 1) - (0)$
$= 18$

(d) Area of blue glass is $18 \times 4 = 72$



Area of rectangle $= 108$

$\therefore$ Area of red glass $= 108 - 72 = 36\,\text{cm}^2$